WEAK SOLUTIONS TO THE FISHER–KOLMOGOROV EQUATION

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1. A crash course in Functional Analysis

1.1. **Spaces.** Let $U \subset \mathbb{R}^n$ be open, and $1 \leq p \leq \infty$. The space $L^p(U)$ is the set of (equivalence classes of) Lebesgue-measurable functions for which the norm

$$\|u\|_{L^p} = \begin{cases} \left(\int_U |u|^p \, \mathrm{d}x \right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty \\ \operatorname{ess\,sup}_{x \in U} |u(x)| & \text{for } p = \infty \end{cases} \tag{1}$$

is finite. (Two functions are identified if they differ only on a set of measure zero.) It can be shown that L^p is a Banach space, i.e. it is complete. Moreover, L^2 is a Hilbert space with inner product

$$\langle u, v \rangle = \int_U u(x) \, v(x) \, \mathrm{d}x \, . \tag{2}$$

(We consider only real-valued functions here; in the complex-valued case, we have to conjugate one of the factors of the integrand.)

For r integer, the Sobolev space $H^r(U)$ is the space of functions whose partial derivatives up to order r are L^2. When $U = \mathbb{T}^n$, it is easy to define H^r for any real r by finiteness of the norm

$$\|u\|_{H^r}^2 = \sum_{k \in \mathbb{Z}^n} (1 + |k|^2)^r \, |\hat{u}_k|^2 \, , \tag{3}$$

where $\hat{u}_k$ denote the Fourier coefficients of u. For $U = \mathbb{R}^n$, a similar construction using the Fourier transform is possible.

If $u = u(x,t)$, we can also think of u as a Banach space valued map $t \mapsto u(\cdot, t)$. Then $u \in L^p([0,T]; X)$ if

$$\|u\|_{L^p([0,T];X)}^p \equiv \int_0^T \|u(t)\|_X^p \, \mathrm{d}t < \infty \, ; \tag{4}$$

similarly, $u \in C([0,T]; X)$ if the map $u \colon [0,T] \to X$, $t \mapsto u(\cdot, t)$ is continuous.

1.2. **Inequalities.** Two basic inequalitites for working with L^p spaces are as simple as they are useful.

Theorem 1 (Young's inequality)**.** *For any two real numbers $a, b > 0$ and for $1 < p, p' < \infty$ with*

$$\frac{1}{p} + \frac{1}{p'} = 1 \tag{5}$$

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(sometimes called the conjugate exponents*),*

$$a\,b \leq \frac{1}{p}\,a^p + \frac{1}{p'}\,b^{p'} \,. \tag{6}$$

Proof. Since the exponential function is convex,

$$a\,b = \mathrm{e}^{\ln a + \ln b} = \mathrm{e}^{\frac{1}{p}\ln a^p + \frac{1}{p'}\ln b^{p'}} \leq \frac{1}{p}\,\mathrm{e}^{\ln a^p} + \frac{1}{p'}\,\mathrm{e}^{\ln b^{p'}} \,. \tag{7}$$

□

Theorem 2 (Hölder)**.** *For $u \in L^p(U)$ and $v \in L^{p'}(U)$ with p and p' conjugate in the sense of* (5)*,*

$$|\langle u, v\rangle| \leq \|u\|_{L^p}\,\|v\|_{L^{p'}} \,. \tag{8}$$

Proof. Since the Hölder inequality is scale invariant, it is sufficient to prove it for functions of norm one in their respective spaces. Thus, taking the absolute value inside the left hand integral and applying the Young inequality, we find that

$$\int_U |u(x)\,v(x)|\,\mathrm{d}x \leq \frac{1}{p}\int_U |u|^p\,\mathrm{d}x + \frac{1}{p'}\int_U |v|^{p'}\,\mathrm{d}x = \frac{1}{p} + \frac{1}{p'} = 1\,. \tag{9}$$

□

1.3. **Projectors.** When $U = \mathbb{T}^n$, any function $u \in L^2$ has the Fourier representation

$$u(x) = \sum_{k \in \mathbb{Z}^n} \hat{u}_k\,\mathrm{e}^{\mathrm{i}k\cdot x} \,, \tag{10}$$

where the functions $\exp(\mathrm{i}kx)$ are an L^2-orthogonal basis. We denote the projection onto the subspace spanned by the finitely many "Fourier modes" up to wave number m by

$$\mathbb{P}_m u(x) = \sum_{|k| \leq m} \hat{u}_k\,\mathrm{e}^{\mathrm{i}k\cdot x} \,. \tag{11}$$

We abbreviate this subspace by $\mathbb{P}_m L^2$. It is easy to check that $\mathbb{P}_m$ is selfadjoint, i.e.

$$\langle u, \mathbb{P}_m v\rangle = \langle \mathbb{P}_m u, \mathbb{P}_m v\rangle = \langle \mathbb{P}_m u, v\rangle \tag{12}$$

for all $u, v \in L^2$.

1.4. **Weak topology.** If X is a Banach space, its dual X' is the set of continuous linear functionals on X, i.e.

$$X' = \{f \in C(X;\mathbb{R}) \colon f \text{ is linear}\} \,. \tag{13}$$

If X is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, the Riesz representation theorem states that $f \in X'$ if and only if there exists a $v \in X$ such that $f(u) = \langle u, v\rangle$ for every $u \in X$. In other words, there is a canonical isomorphism between X and its dual, and we will henceforth interchange the notion of function and functional with impunity.

Similarly, it can be shown that the dual of L^p is L^q for $1/p + 1/q = 1$ with $1 \leq p < \infty$, where a function $\phi \in L^q$ is identified with the functional $\langle\cdot,\phi\rangle \in (L^p)'$ via (an extension of) the L^2 inner product.

If X is a Banach space, a sequence $u_n \in X$ is said to converge weakly to $u \in X$ if

$$\langle u_n, v\rangle \to \langle u, v\rangle \tag{14}$$

for every $v \in X'$; we write $u_n \rightharpoonup u$. This defines a topology on X which is weaker than the norm topology (or strong topology) on X unless X is finite dimensional, in which case weak and strong topology coincide. In other words, if X is infinite dimensional, there are sequences that converge weakly, but not strongly.

The following theorem is a central ingredient for our construction. We formulate it for the special case of a reflexive Banach space, i.e. a space where $X = X''$. This is, for example, the case for L^p with $1 < p < \infty$. Both parts of the theorem are consequences of fundamental, nontrivial theorems of functional analysis: the uniform boundedness principle and Alaoglu's theorem.

Theorem 3. *Let X be a reflexive Banach space and u_n a sequence in X. Then*

(1) *if u_n is weakly convergent, then it is bounded in the norm of X;*
(2) *vice versa, if u_n is bounded, then the set $\{u_n\}$ is relatively compact in the weak topology, i.e. there exists a weakly convergent subsequence of u_n.*

We can also endow the spaces for functions of space and time with various weak topologies. Given a Banach space X with dual X', we say that $u_n \colon [0,T] \to X$ converges to u in $C([0,T]; \text{w-}X)$, where w-X denotes X endowed with its weak topology, when for every $v \in X'$,

$$\langle u_n(t), v\rangle \to \langle u(t), v\rangle \tag{15}$$

uniformly on $[0,T]$. Similarly, $u_n \to u$ in w-$L^p([0,T]; \text{w-}X)$ if

$$\int_0^T \langle u_n(t), v(t)\rangle \,\mathrm{d}t \to \int_0^T \langle u(t), v(t)\rangle \,\mathrm{d}t\,, \tag{16}$$

for every $v \in L^{p'}([0,T]; X')$ with $1 \le p < \infty$ and $p' = p/(p-1)$ the Hölder conjugate of p.

1.5. **Compactness theorems.**

Theorem 4 (Rellich). *The embedding $H^t(\mathbb{T}) \hookrightarrow H^s(\mathbb{T})$ is compact for all real numbers $s < t$.*

This statement is equivalent to saying that if $u_n \rightharpoonup u$ weakly in $H^t(\mathbb{T})$, then $u_n \to u$ strongly in $H^s(\mathbb{T})$.

Proof. Assume that $u_n \rightharpoonup 0$ weakly in $H^t(\mathbb{T})$. It is sufficient to prove that this implies $u_n \to 0$ strongly in $H^s(\mathbb{T})$. Let $\varepsilon > 0$, and estimate

$$\begin{aligned}
\|u_n\|_{H^s}^2 &= \sum_{|k|\le\kappa} (1+|k|^2)^s\, |\hat u_k|^2 + \sum_{|k|>\kappa} (1+|k|^2)^{s-t}\, (1+|k|^2)^t\, |\hat u_k|^2 \\
&\le \sum_{|k|\le\kappa} (1+|k|^2)^s\, |\hat u_k|^2 + (1+|\kappa|^2)^{s-t}\, \|u_n\|_{H^t}^2 \\
&\le \sum_{|k|\le\kappa} (1+|k|^2)^s\, |\hat u_k|^2 + \frac{\varepsilon}{2}
\end{aligned} \tag{17}$$

provided that κ is chosen large enough. This choice can be made independent of n because the weak convergence in H^t implies that the H^t-norm of u_n is bounded independent of n. The first term on the right of (17) is a norm on the finite dimensional subspace $\mathbb{P}_\kappa L^2(\mathbb{T})$ where weak and strong topology coincide. Thus, this term can also be made less than $\frac{\varepsilon}{2}$ choosing n sufficiently large. □

Corollary 5. *For every $T > 0$ the embedding*

$$L^2([0,T]; L^2(\mathbb{T})) \hookrightarrow C([0,T]; \text{w-}L^2(\mathbb{T})) \cap \text{w-}L^2([0,T]; \text{w-}H^1(\mathbb{T})) \,, \tag{18}$$

is continuous, where the intersection on the right is endowed with the relative topology induced by the inclusion map. (In other words, a sequence converges in the intersection iff it converges in each of the spaces separately.)

Proof. Let u_n be a sequence in the intersection on the right, converging to zero in $C([0,T]; \text{w-}L^2(\mathbb{T}))$ as well as in $\text{w-}L^2([0,T]; \text{w-}H^1(\mathbb{T}))$. Let $\varepsilon > 0$. Then, using the inequality

$$\|v\|_{L^2}^2 \leq \|v\|_{H^{-1}} \|v\|_{H^1} \tag{19}$$

which is a direct consequence of the Cauchy-Schwarz inequality in the Fourier representation, followed by a Young inequality, we estimate

$$\begin{aligned} \int_0^T \|u_n\|_{L^2}^2 \, \mathrm{d}t &\leq \int_0^T \|u_n\|_{H^{-1}} \|u_n\|_{H^1} \, \mathrm{d}t \\ &\leq \frac{1}{2\delta} \int_0^T \|u_n\|_{H^{-1}}^2 \, \mathrm{d}t + \frac{\delta}{2} \int_0^T \|u_n\|_{H^1}^2 \, \mathrm{d}t \\ &\leq \frac{1}{2\delta} \int_0^T \|u_n\|_{H^{-1}}^2 \, \mathrm{d}t + \frac{\varepsilon}{2} \end{aligned} \tag{20}$$

provided that δ is chosen small enough. This choice can be made independent of n because the weak convergence in $\text{w-}L^2([0,T]; \text{w-}H^1(\mathbb{T}))$ implies boundedness with respect to the norm. Moreover, convergence in $C([0,T]; \text{w-}L^2(\mathbb{T}))$ implies, by the Rellich theorem, convergence in $C([0,T]; H^{-1}(\mathbb{T}))$. Thus, the first term on the right of (20) can also be made less than $\varepsilon/2$ choosing n sufficiently large. □

The following theorem will be needed later to lift the preceding L^2-type continuity result into L^p spaces of higher index.

Theorem 6. *If $1 < p < q < r < \infty$, the embedding*

$$L^q(U) \hookrightarrow L^p(U) \cap \text{w-}L^r(U) \tag{21}$$

is continuous.

This statement is equivalent to saying that if $u_n \to u$ strongly in L^p and $u_n \rightharpoonup u$ weakly in L^r, then $u_n \to u$ strongly in any intermediate L^q.

Proof. As before, we only need to consider a sequence $u_n \to 0$ strongly in L^q and $u_n \rightharpoonup 0$ weakly in L^r. Let $\varepsilon > 0$. By the Young inequality, for every $\delta > 0$ and pairs of positive conjugate exponents satisfying

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{t} + \frac{1}{t'} = 1 \,, \tag{22}$$

there exists a constant $c(\delta)$ such that

$$|u_n|^q = |u_n|^{\frac{q}{s}} \, |u_n|^{\frac{q}{s'}} \leq c(\delta) \, |u_n|^{\frac{tq}{s}} + \delta \, |u_n|^{\frac{t'q}{s'}} \,. \tag{23}$$

Choosing

$$t = \frac{r-p}{r-q} \quad \text{and} \quad s = \frac{q}{p}\,\frac{r-p}{r-q} \tag{24}$$

we first observe that, due to the condition $1 < p < q < r < \infty$, all conjugate exponents s, s', t, and t' are positive and larger than 1. We can thus proceed and conclude that

$$\int_U |u_n|^q \,\mathrm{d}x \leq c(\delta) \int_U |u_n|^p \,\mathrm{d}x + \delta \int_U |u_n|^r \,\mathrm{d}x \tag{25}$$

Since u_n is weakly convergent in L^r, it is, in particular, bounded in the L^r norm independent of n. By choosing δ sufficiently small, we can ensure that the second term on the right is less than $\varepsilon/2$. Moreover, $u_n \to 0$ in L^p, so that, for some N large enough, the first term is also less than $\varepsilon/2$ for every $n \geq N$. We conclude that $u_n \to 0$ strongly in L^q. □

The following is a weak version of the Arzelà–Ascoli theorem, which we state here without proof.

Theorem 7 (Arzelà–Ascoli)**.** *Let X be a Banach space, and let $u_n \in C([0,T]; \text{w-}X)$ be a sequence of functions. Assume that*

(i) *$\{u_n\}$ is pointwise bounded, i.e. for every $t \in [0,T]$ the set $\{u_n(t)\}$ is bounded;*
(ii) *$\{u_n\}$ is weakly equicontinuous, i.e. for every $\psi \in X'$ the set $\langle \psi, u_n \rangle$ is equicontinuous.*

Then $\{u_n\}$ is relatively compact in $C([0,T]; \text{w-}X)$, i.e. there exists a subsequence, still denoted u_n, and a function $u \in C([0,T]; \text{w-}X)$, so that

$$\sup_{0 \leq t \leq T} |\langle u_n - u, \psi \rangle| \to 0 \tag{26}$$

as $n \to \infty$ for every $\psi \in X'$.

2. The Fisher–Kolmogorov equation

The Fisher–Kolmogorov equation,

$$\partial_t u = \partial_{xx} u + (1-u)\, u^m \,, \tag{27a}$$

$$u(0) = u^{\text{in}} \,, \tag{27b}$$

where m is an even positive integer, models the concentration of the autocatalyst in an autocatalytic chemical reaction of order $m+1$.

Let us suppose for the moment that the Fisher–Kolmogorov possesses a smooth solution. We multiply the equation with a test function $\psi \in H^1(\mathbb{T})$ and integrate in space and time, obtaining

$$\langle u(t_2), \psi \rangle - \langle u(t_1), \psi \rangle = -\int_{t_1}^{t_2} \langle u_x, \psi_x \rangle \,\mathrm{d}t + \int_{t_1}^{t_2} \langle (1-u)\, u^m, \psi \rangle \,\mathrm{d}t \,. \tag{28}$$

This expression makes sense even if smooth solutions to (27) do not exist. This motivates the following definition.

Definition 8. *We say that u solves the weak Fisher–Kolmogorov equation, if*

$$u \in C([0,T]; L^2(\mathbb{T})) \cap L^2([0,T]; H^1(\mathbb{T})) \cap L^{m+2}([0,T]; L^{m+2}(\mathbb{T})) \,, \tag{29}$$

and u satisfies (28) *for every $\psi \in H^1(\mathbb{T})$ and every $[t_1, t_2] \subset [0,T]$.*

3. Existence of weak solutions

We construct the solution u to the weak form of the Fisher–Kolmogorov equation from a sequence $\{u_n\}$ of solutions to an approximate, regularized system in the limit of vanishing regularization. There are many different ways to regularize a PDE, and an effective choice is not always obvious. For equations that are posed on the torus, it is easiest to project the system onto a finite number of Fourier modes. More precisely, we apply the projector $\mathbb{P}_n$ to each of (27),

$$\partial_t u_n = \partial_{xx} u_n + \mathbb{P}_n((1-u_n)\, u_n^m)\,, \tag{30a}$$

$$u_n(0) = \mathbb{P}_n u^{\text{in}}\,. \tag{30b}$$

Note that in general $u_n(t) \neq \mathbb{P}_n u(t)$, even though this is true initially. The reason is that projection does not commute with multiplication, so that the projector on the last term of (30a) cannot be removed, and the regularized evolution equation differs from the original equation. As any function in L^2 can be approximated by functions with a finite number of nonvanishing Fourier coefficients, we hope to recover a solution to the full Fisher–Kolmogorov equation in the limit $n \to \infty$.

3.1. Existence of the regularized solution.

Note that (30) is an ordinary differential equation with Lipshitz right hand side. This is immediate by expanding u_n as a (finite!) Fourier sum. Differentiation in x corresponds to multiplication by constants in the Fourier representation, multiplication corresponds to convolution, and the projection corresponds to dropping some terms from this convolution. Thus, in the Fourier representation, the right hand side of (30a) is just a (messy) polynomial in the Fourier coefficients, hence satisfies a Lipshitz condition.

The classical Picard existence theory for ordinary differential equations (which is not the topic of these notes, but can be found in most books on differential equations), asserts that, for every fixed n, there exists a time T_n so that (30) has a solution

$$u_n \in C^1([0,T_n); \mathbb{P}_n L^2(\mathbb{T}))\,. \tag{31}$$

Moreover, if T_n denotes the supremum over all times for which such solution exists, then $T_n < \infty$ only if $\|u_n(t)\| \to \infty$ as $t \to T_n$, where $\|\cdot\|$ denotes any norm on the finite dimensional vector space $\mathbb{P}_n L^2(\mathbb{T})$.

In other words, a solution to our regularized equation (30) can only cease to exist by blowing up. In the following, we show that blow-up does not occur. However, (31) gives us the right to perform the necessary manipulations—initially only up to T_n, then, by extension, up to $t = \infty$.

3.2. *A priori* estimates.

The term "*a priori* estimates" in the context of partial differential equations refers to estimates on the solution that are performed under the yet unproven assumption that a solution exists, and that this solution is smooth enough to make all the steps in the derivation of the estimate true. Usually, finding good *a priori* estimates is the key difficulty in proving existence of solutions, often so much so that authors publish only the estimates, but do not bother to write down full proofs.

Turning an *a priori* estimate into a proof works by proving that the same estimate holds for the regularized system (which is often, but not always, true), and passing to the limit. In the following, we will work with the regularized solution, known to exist at least for a short while, from the start.

Multiplying (30a) with u_n and integrating in space, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} u_n^2\,\mathrm{d}x = -\int_{\mathbb{T}} |\partial_x u_n|^2\,\mathrm{d}x + \int_{\mathbb{T}} u_n^{m+1}\mathrm{d}x - \int_{\mathbb{T}} u_n^{m+2}\mathrm{d}x\,. \tag{32}$$

Note that the torus does not have a boundary—hence no boundary terms when integrating by parts. Moreover, the projector $\mathbb{P}_n$ is self-adjoint with respect to the L^2 inner product, so that

$$\int u_n\,\mathbb{P}_n u_n^m\,\mathrm{d}x = \int \mathbb{P}_n u_n\,u_n^m\,\mathrm{d}x = \int u_n\,u_n^m\,\mathrm{d}x\,; \tag{33}$$

similarly for the last term in (32).

Using the Hölder and Young inequalities, we find that

$$\begin{aligned}\int_{\mathbb{T}} u^{m+1}\,\mathrm{d}x &= \int_{\mathbb{T}} u^{\frac{m+1}{p}}\,u^{\frac{m+1}{p'}}\,\mathrm{d}x \\ &\le \left(\int_{\mathbb{T}} u^{\frac{m+1}{p}q}\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{\mathbb{T}} u^{\frac{m+1}{p'}q'}\,\mathrm{d}x\right)^{\frac{1}{q'}} \\ &\le \frac{1}{q}\int_{\mathbb{T}} u^{\frac{m+1}{p}q}\,\mathrm{d}x + \frac{1}{q'}\int_{\mathbb{T}} u^{\frac{m+1}{p'}q'}\,\mathrm{d}x\,, \end{aligned}\tag{34}$$

where

$$\frac{1}{p}+\frac{1}{p'}=1 \quad\text{and}\quad \frac{1}{q}+\frac{1}{q'}=1\,. \tag{35}$$

When applying this estimate to (32), we need to "interpolate" the intermediate term of indefinite sign between the negative definite term of higest order, and the lowest order term. Thus, we demand that

$$\frac{m+1}{p}\,q=2 \quad\text{and}\quad \frac{m+1}{p'}\,q'=m+2\,, \tag{36}$$

which implies, after some simple algebra, that

$$p=\frac{m(m+1)}{2}\,,\quad q=m\,,\quad p'=\frac{m(m+1)}{m(m+1)-2}\,,\quad\text{and}\quad q'=\frac{m}{m-1}\,. \tag{37}$$

Inserting this estimate into (32), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} u_n^2\,\mathrm{d}x \le -\int_{\mathbb{T}} |\partial_x u_n|^2\,\mathrm{d}x + \frac{1}{m}\int_{\mathbb{T}} u_n^2\,\mathrm{d}x - \frac{1}{m}\int_{\mathbb{T}} u_n^{m+2}\,\mathrm{d}x\,. \tag{38}$$

Dropping the negative definite terms from the right, we obtain the differential inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} u_n^2\,\mathrm{d}x \le c\int_{\mathbb{T}} u_n^2\,\mathrm{d}x\,, \tag{39}$$

which, upon integration in time, proves that

$$\int_{\mathbb{T}} u_n^2(x,t)\,\mathrm{d}x \le \mathrm{e}^{ct}\int_{\mathbb{T}} u_n^2(x,0)\,\mathrm{d}x = \mathrm{e}^{ct}\int_{\mathbb{T}} (\mathbb{P}_n u^{\mathrm{in}})^2\,\mathrm{d}x \le \mathrm{e}^{ct}\int_{\mathbb{T}} (u^{\mathrm{in}})^2\,\mathrm{d}x\,. \tag{40}$$

This estimate proves that u_n cannot blow up in finite time; the time interval of existence asserted in Section 3.1 is therefore $T_n=\infty$. Moreover, the estimate is uniform in n; it is clearly not uniform in t, but this is does not matter. We have therefore shown that

$$\{u_n\} \text{ is bounded in } L^\infty([0,T];L^2(\mathbb{T})) \tag{41}$$

for any fixed $T>0$.

Integrating (38) in time and rearranging terms, we obtain

$$\int_0^T \int_{\mathbb{T}} |\partial_x u_n|^2 \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{m} \int_0^T \int_{\mathbb{T}} u_n^{m+2} \, \mathrm{d}x \, \mathrm{d}t \\ \leq \int_{\mathbb{T}} u_n^2(x,0) \, \mathrm{d}x - \int_{\mathbb{T}} u_n^2(x,T) \, \mathrm{d}x + \frac{1}{m} \int_0^T \int_{\mathbb{T}} u_n^2 \, \mathrm{d}x \, \mathrm{d}t \, . \quad (42)$$

All the terms on the right are bounded by (41) independent of n. Since each term on the left is non-negative, each must be bounded independent of n as well. In other words,

$$\{u_n\} \text{ is bounded in } L^2([0,T]; H^1(\mathbb{T})) \quad (43)$$

and

$$\{u_n\} \text{ is bounded in } L^{m+2}([0,T]; L^{m+2}(\mathbb{T})) \, . \quad (44)$$

Boundedness of a sequence in a Banach space implies the existence of a weakly convergent subsequence. However, weak convergence is not sufficient to assert that the limit will actually solve the full Fisher–Kolmogorov equation. Thus, we need so-called *compactness results* to strengthen the notion of convergence of our approximating sequence.

3.3. **Compactness theorems.** We apply the Arzelà–Ascoli theorem, Theorem 7, to our approximating sequence $\{u_n\}$ with $X = L^2(\mathbb{T})$. Pointwise boundedness has already been proved, as stated in (41). To prove weak equicontinuity, we integrate (30) in time, multiply with a test function $\psi_k \in \mathbb{P}_k L^2(\mathbb{T})$ for some fixed k, and integrate in space, so that

$$\int_{\mathbb{T}} \psi_k(x) \, u_n(x,t_2) \, \mathrm{d}x - \int_{\mathbb{T}} \psi_k(x) \, u_n(x,t_1) \, \mathrm{d}x \\ = \int_{t_1}^{t_2} \int_{\mathbb{T}} \psi_k \, \partial_{xx} u_n \, \mathrm{d}x \, \mathrm{d}t + \int_{t_1}^{t_2} \int_{\mathbb{T}} \psi_k \, \mathbb{P}_n((1-u_n) \, u_n^m) \, \mathrm{d}x \, \mathrm{d}t \, . \quad (45)$$

Without loss of generality, assume that $n \geq k$, so that the projector in the last term is redundant. Integrating by parts in the first term on the right, taking absolute values, and applying the Cauchy–Schwarz and Hölder inequalities, we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{T}} \psi_k(x)\, u_n(x,t_2)\,\mathrm{d}x - \int_{\mathbb{T}} \psi_k(x)\, u_n(x,t_1)\,\mathrm{d}x \right| \\
&\quad \le \int_{t_1}^{t_2} \int_{\mathbb{T}} |\partial_x \psi_k|\, |\partial_x u_n| \,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2} \int_{\mathbb{T}} |\psi_k|\, |u_n^m| \,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2} \int_{\mathbb{T}} |\psi_k|\, |u_n^{m+1}| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \|\partial_x \psi_k\|_{L^\infty} \left(\int_{t_1}^{t_2} \int_{\mathbb{T}} 1^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \int_{\mathbb{T}} |\partial_x u_n|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\qquad + \|\psi_k\|_{L^\infty} \left(\int_{t_1}^{t_2} \int_{\mathbb{T}} \mathrm{d}x\,\mathrm{d}t \right)^{\frac{2}{m+2}} \left(\int_{t_1}^{t_2} \int_{\mathbb{T}} u_n^{m+2} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{m}{m+2}} \\
&\qquad + \|\psi_k\|_{L^\infty} \left(\int_{t_1}^{t_2} \int_{\mathbb{T}} \mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{m+2}} \left(\int_{t_1}^{t_2} \int_{\mathbb{T}} u_n^{m+2} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{m+1}{m+2}} \\
&\quad \le c(\psi_k)\, |t_2 - t_1|^{\frac{1}{2}}\, \|u_n\|_{L^2([0,T];H^1(\mathbb{T}))} \\
&\qquad\quad + c(\psi_k)\, |t_2 - t_1|^{\frac{2}{m+2}}\, \|u_n\|^m_{L^{m+2}([0,T];L^{m+2}(\mathbb{T}))} \\
&\qquad\quad + c(\psi_k)\, |t_2 - t_1|^{\frac{1}{m+2}}\, \|u_n\|^{m+1}_{L^{m+2}([0,T];L^{m+2}(\mathbb{T}))} \,.
\end{aligned} \tag{46}$$

Since $\{u_n\}$ is bounded in the norms indicated, as stated in (43) and (44), nothing on the right of (46) depends on n, and we have proved weak equicontinuity with test functions from $\mathbb{P}_k L^2(\mathbb{T})$.

We extend the estimate to an arbitrary test function $\psi \in L^2(\mathbb{T})$ by density as follows. Let $\varepsilon > 0$ fixed. Since $\psi_k \equiv \mathbb{P}_k \psi \to \psi$ as $k \to \infty$, there exists some k such that

$$\|\psi - \psi_k\|_{L^2(\mathbb{T})} \le \frac{\varepsilon}{3K}\,, \tag{47}$$

where K is a bound on $\{u_n\}$ in $L^\infty([0,T];L^2(\mathbb{T}))$. For this value of k, choose $t_2 - t_1$ small enough such that the right side of (46) is less than $\varepsilon/3$. Then

$$\begin{aligned}
&\left| \int_{\mathbb{T}} \psi(x)\, u_n(x,t_2)\,\mathrm{d}x - \int_{\mathbb{T}} \psi(x)\, u_n(x,t_1)\,\mathrm{d}x \right| \\
&\quad \le \left| \int_{\mathbb{T}} (\psi(x) - \psi_k(x)\, u_n(x,t_2)\,\mathrm{d}x \right| + \left| \int_{\mathbb{T}} \psi_k(x)\, (u_n(x,t_2) - u_n(x,t_1))\,\mathrm{d}x \right| \\
&\qquad + \left| \int_{\mathbb{T}} (\psi_k(x) - \psi(x)\, u_n(x,t_1)\,\mathrm{d}x \right| \\
&\quad \le \|\psi - \psi_k\|_{L^2(\mathbb{T})}\, \|u_n\|_{L^\infty([0,T];L^2(\mathbb{T}))} + \frac{\varepsilon}{3} + \|\psi - \psi_k\|_{L^2(\mathbb{T})}\, \|u_n\|_{L^\infty([0,T];L^2(\mathbb{T}))} \\
&\quad \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,.
\end{aligned} \tag{48}$$

This completes the proof of weak equicontinuity for $\{u_n\}$.

In conclusion, we have found that the sequence $\{u_n\}$ is relatively compact in $C([0,T];\text{w-}L^2(\mathbb{T}))$. It is also relatively compact in $\text{w-}L^2([0,T];\text{w-}H^1(\mathbb{T}))$ due to (43). Hence, by the Rellich compactness theorem in the form of Corollary 5, we conclude that $\{u_n\}$ is relatively compact in $L^2([0,T];L^2(\mathbb{T}))$.

3.4. **Passing to the limit.** We now have all the ingredients to conclude the existence of a limit function u in the following sense. Since $\{u_n\}$ is bounded in $L^2([0,T];H^1(\mathbb{T}))$, there exists weakly converging subsequence which, for simplicity, we still denote by u_n. (One can actually show with little extra effort that the entire sequence converges—this is of interest when using the approximate system as a numerical method.) A similar argument follows from boundedness in $L^{m+2}([0,T];L^{m+2}(\mathbb{T}))$, so that

$$u_n \rightharpoonup u \quad \text{in w-}L^2([0,T];\text{w-}H^1(\mathbb{T}))\,, \tag{49a}$$

$$u_n \rightharpoonup u \quad \text{in w-}L^{m+2}([0,T];\text{w-}L^{m+2}(\mathbb{T}))\,. \tag{49b}$$

Moreover, the conclusion of the previous section implies that (strictly speaking once again by extracting subsequences)

$$u_n \to u \quad \text{in } L^2([0,T];L^2(\mathbb{T}))\,, \tag{49c}$$

$$u_n \to u \quad \text{in } C([0,T];\text{w-}L^2(\mathbb{T}))\,. \tag{49d}$$

Note that the limit function does not depend on the topology. This can ben seen as follows. The w-$L^2([0,T];\text{w-}L^2(\mathbb{T}))$ topology is weaker than each of stated topologies, but is still Hausdorff, i.e. it separates points. Thus, after passing to a common subsequence, each of the four convergence statements can be weakened to convergence in w-$L^2([0,T];\text{w-}L^2(\mathbb{T}))$, so that the unique limit function is also common to all.

However, the existence of a limit does not guarantee that this limit solves the full Fisher–Kolmogorov equation in any sense. Obviously, the limit we have achieved so far is not even twice continuously differentiable—thus we cannot expect that (27) is satisfied in the classical sense—but matches our requirements in the definition of a weak solution perfectly.

Since the approximate system (30) is posed on a finite dimensional subspace of L^2, its weak and strong formulations are equivalent. We will now show that each term in the weak formulation of (30) converges to the corresponding term in (28).

Fix $\psi \in H^1(\mathbb{T})$. First, (49d) implies that

$$\langle u_n(t), \psi \rangle \to \langle u(t), \psi \rangle \tag{50}$$

for every $t \in [0,T]$. Hence, the two terms on the left of (28) are recovered as $n \to \infty$.

Next, (49a) implies that

$$\int_{t_1}^{t_2} \langle \partial_x u_n, \partial_x \psi \rangle \, \mathrm{d}t \to \int_{t_1}^{t_2} \langle \partial_x u, \partial_x \psi \rangle \, \mathrm{d}t\,. \tag{51}$$

(Note that we are again using the L^2 inner product to represent functionals on H^1.)

To prove convergence in the nonlinear term, we note that, by the mean value theorem,

$$|u_n^m - u^m| \le m\,|u_n - u|\left(|u_n|^{m-1} + |u|^{m-1}\right). \tag{52}$$

Hence,

$$\begin{aligned}\int_{t_1}^{t_2} \langle u_n^m - u^m, \psi \rangle \, \mathrm{d}t &\le m\,\|\psi\|_{L^\infty} \int_{t_1}^{t_2} \int_{\mathbb{T}} |u_n - u|\left(|u_n|^{m-1} + |u|^{m-1}\right) \mathrm{d}x\,\mathrm{d}t \\ &\le m\,\|\psi\|_{L^\infty}\,\|u_n - u\|_{L^{(m+2)/3}}\left(\|u_n\|_{L^{m+2}}^{m-1} + \|u\|_{L^{m+2}}^{m-1}\right).\end{aligned} \tag{53}$$

The terms in parentheses are bounded by (44). Moreover, $u_n \to u$ in $L^{(m+2)/3}$. When $(m+2)/3 \leq 2$, this is a direct consequence of (49c); when $2 < (m+2)/3 < m+2$, this results from applying Theorem 6 to (49b) and (49c).

A similar argument can be made for weak convergence of the difference $u_n^{m+1} - u^{m+1}$, so that we conclude that

$$\int_{t_1}^{t_2} \langle \mathbb{P}_n((1-u_n)\,u_n^m), \psi \rangle \, \mathrm{d}t \to \int_{t_1}^{t_2} \langle (1-u)\,u^m, \psi \rangle \, \mathrm{d}t \tag{54}$$

as $n \to \infty$ for every $\psi \in \mathbb{P}_k L^2(\mathbb{T})$ with k fixed; hence, by density, also for every $\psi \in H^1(\mathbb{T})$. This completes the proof.