Partial Differential Equations

Midterm Exam

October 29, 2008

1. Solve the partial differential equation

$$x\,u_t - t\,u_x = 0\,,$$

where $u = u(x,t)$ for $(x,t) \in \mathbb{R}^2 \setminus (0,0)$. What are the characteristic curves? What kind of initial/boundary data do you need to prescribe?

Hint: Start out with an ansatz for the charactistic curves of the general form $x = x(s)$ and $t = t(s)$. (10)

2. Show that if u is harmonic on some open $U \subset \mathbb{R}^n$, then u cannot have an isolated zero in U. (5)

3. Let $U \in \mathbb{R}^n$ be open and bounded. State a set of boundary conditions which are sufficient to guarantee that a solution $u \in C^4(\bar{U})$ of the Poisson-type problem for the bi-Laplacian,

$$\Delta^2 u = f\,,$$

satisfying those boundary conditions, is unique.

Hint: Energy methods. (10)

4. Consider the following initial-boundary value problem (IBVP) for the heat equation on $U = (-\pi/2, \pi/2)$,

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } U \times (0,\infty)\,, \\ u &= 0 && \text{on } \{x = \pm\pi/2\} \times (0,\infty)\,, \\ u &= g && \text{on } U \times \{t = 0\}\,. \end{aligned}$$

(a) Let $u_i \in C^2_1(U \times (0,\infty))$ solve the IBVP with initial data $g_i \in C(\bar{U})$ for $i = 1,2$. Show that if $g_1 \le g_2$, then $u_1 \le u_2$ for all $(x,t) \in \bar{U} \times [0,\infty)$.

Note: You may quote a well-known theorem from class; no need to prove from scratch.

(b) Show that the IBVP has a particular solutions of the form

$$u(x,t) = a(t)\cos x\,.$$

Derive an expression for $a(t)$.

(c) Conclude that

$$|u(x,t)| \leq c\, e^{-t}$$

where c depends only on g.

Note: For simplicity of the argument, you may assume that g is compactly supported in U.

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5. Let $f \in C^2(\mathbb{R})$ and $b \in \mathbb{R}^n$. Show that $f(b \cdot x - t)$ solves the wave equation in $\mathbb{R}^n$. Describe the geometry of the solution. What is the speed of propagation? (10)