

# Partial Differential Equations

## Homework 2

due September 19, 2008

1. (a) The *standard mollifier* is defined by

$$\eta(x) \equiv \begin{cases} c(n) \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where  $c(n)$  is chosen such that

$$\int_{\mathbb{R}^n} \eta(x) dx = 1.$$

Show that  $\eta \in C^\infty(\mathbb{R}^n)$ .

- (b) Show that if  $\eta_\varepsilon$  is a radial mollifier, and  $u$  is a radial, locally integrable function, then its mollification

$$u_\varepsilon(x) = (\eta_\varepsilon * u)(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(y) u(x - y) dy$$

is also radial.

2. Let  $X \subset \mathbb{R}^n$ . Show that

- (a)  $X$  is connected iff  $\emptyset$  and  $X$  are the only subsets of  $X$  that are both relatively open and relatively closed in  $X$ .
- (b) If  $\{W_\alpha\}_{\alpha \in A}$  is a collection of connected subsets of  $X$  such that

$$\bigcap_{\alpha \in A} W_\alpha \neq \emptyset,$$

then  $\cup_{\alpha \in A} W_\alpha$  is connected.

- (c) If  $X$  is connected, then  $\overline{X}$  is connected.
- (d) Every point  $x \in X$  is contained in a unique maximal connected subset of  $X$ , and this subset is relatively closed in  $X$ .

The relevant definitions from point-set topology in  $\mathbb{R}^n$ :

- $A \subset X$  is called *relatively open in  $X$*  if for every  $x \in A$  there exists an  $\varepsilon > 0$  such that  $X \cap B(x, \varepsilon) \subset A$ .
- $B \subset X$  is called *relatively closed in  $X$*  if  $A = X \setminus B$  is relatively open in  $X$ .
- $X$  is called *disconnected* if there exist disjoint, nonempty subsets  $A_1, A_2 \subset X$  that are relatively open in  $X$  and  $X = A_1 \cup A_2$ .
- $X$  is called *connected* if it is not disconnected.

3. Evans, p. 85 problem 3.

4. Evans, p. 86 problem 4.