Partial Differential Equations

Homework 9

due December 3, 2008

1. Consider the inviscid Burger's equation on $\mathbb{T}$, i.e.

$$u_t + u\, u_x = 0\,.$$

   (a) Define an approximate solution u_n by applying the projector $\mathbb{P}_n$ which projects onto modes up to wave number n to Burger's equation. Show that

$$\|u_n(t)\|_{L^2} = \|u_n(0)\|_{L^2}\,.$$

   (b) Conclude that $\{u_n\}$ has a subsequence that converges to some u weakly in $L^2(\mathbb{T})$, and that

$$\|u(t)\|_{L^2} \le \|u(0)\|_{L^2}\,.$$

   Why do you have an inequality rather than equality?

   Notes: You are not required to show that u solves Burger's equation in any sense. This would require a much more involved analysis. You can use, without proof, the fact that if $x_n \rightharpoonup x$ weakly on a Hilbert space, then

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|\,.$$

2. (From Evans, p. 164, Question 13.) Let $u \in C(\mathbb{R} \times [0,T])$ for some $T > 0$ be a weak solution to the scalar conservation law

$$\begin{aligned} u_t + F(u)_x &= 0 \qquad \text{in } \mathbb{R} \times (0,T)\,,\\ u &= g \qquad \text{on } \mathbb{R} \times \{t = 0\}\,. \end{aligned}$$

   Assume further that for any fixed $t \in [0,T]$, $u(\,\cdot\,,t)$ has compact support in $\mathbb{R}$, and that $F(0) = 0$. Show that

$$\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} g(x)\,dx$$

   for every $t \in [0,T]$.

3. (From Evans, p. 164, Question 14.) Compute explicitly the unique entropy solution of

$$\begin{aligned} u_t + u\,u_x &= 0 \qquad \text{in } \mathbb{R} \times (0,\infty)\,, \\ u &= g \qquad \text{on } \mathbb{R} \times \{t=0\}\,. \end{aligned}$$

for

$$g(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 < x\,. \end{cases}$$

Draw the characteristic curves in an (x,t)-plot, being sure to document all qualitative changes in the solution for $t > 0$.