PDE – Final exam solutions – Fall 2008

1(a) Recall that $D|x| = \frac{x}{|x|}$ and $D\cdot x = n$.

$$Dh = 2x\,|x-x_0|^{-n} + (|x|^2 - R^2)(-n)\frac{x-x_0}{|x-x_0|^{n+2}}$$

$$\Delta h = D\cdot Dh = 2n\,|x-x_0|^{-n} + 2x\cdot(-n)\frac{x-x_0}{|x-x_0|^{n+2}}$$

$$+\ 2x\cdot(-n)\frac{x-x_0}{|x-x_0|^{n+2}} - n(|x|^2-R^2)\frac{n}{|x-x_0|^{n+2}} + n(n+2)(|x|^2-R^2)\frac{1}{|x-x_0|^{n+2}}$$

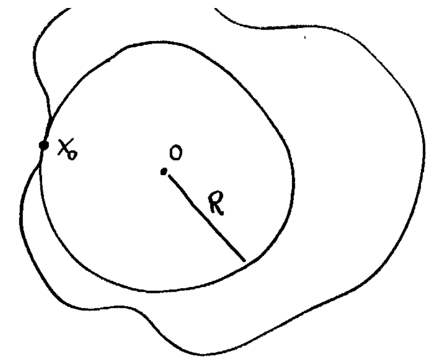
$$= \frac{2n}{|x-x_0|^{n+2}}\underbrace{\left[\,|x-x_0|^2 - 2x\cdot(x-x_0) + (|x|^2-R^2)\right]}_{=0}$$

$$= 0 \quad \text{if } x \neq x_0$$

Note: The first term of h is a multiple of the Poisson kernel for the disk, see Evans.

(b) Choose R such that $B(0,R)$ is the largest open ball centered at the origin contained in U (since U is open, such R exists). Then $\partial B(0,R) \cap \partial U$ is non-empty and we can choose

$$x_0 \in \partial B(0,R) \cap \partial U .$$



now set $u = h$ with x_0 and $R = |x_0|$ as above and note that

$$h(x) > 0 \qquad \text{if} \quad |x| > R \qquad (*)$$

Since h is harmonic in U,

$$0 = h(0) = \fint_U h(x)\,dx$$

$$= \int_U h(x)\,dx$$

$$= \underbrace{\int_{B(0,R)} h(x)\,dx}_{\substack{=|B(0,R)|\,h(0) \\ =0 \\ \text{as } h \text{ is harmonic} \\ \text{on } B(0,R)}} + \int_{U\setminus B(0,R)} h(x)\,dx$$

so the last term is strictly positive due to $(*)$ unless $U = B(0,R)$. $\square$

2. Let u, v be two such solutions and set $w = u - v$. Then

$$w_t = \Delta w - u^2 + v^2$$
$$= \Delta w - w(u+v)$$

Now multiply with w and integrate:

$$\int_U w w_t = \int_U w \Delta w - \int_U w^2 (u+v)$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_U w^2 \, dx = -\int_U |Dw|^2 \, dx - \int_U w^2 (u+v)\, dx \quad \text{as } w=0 \text{ on } \partial U$$

$$\Rightarrow \frac{d}{dt}\int_U w^2 \, dx \leq \underbrace{2\left(\|u\|_{L^\infty(U_T)} + \|v\|_{L^\infty(U_T)}\right)}_{\leq c < \infty} \int_U w^2 \, dx$$

$$\Rightarrow \int_U w^2(t)\, dx \leq \underbrace{\int_U w^2(0)\, dx}_{=0} \; e^{cT}$$

$$\Rightarrow w \equiv 0 \text{ on } U_T.$$

3(a) This can be verified by direct differentiation; setting $t=0$ in the expressions for u and u_t recovers the initial conditions.

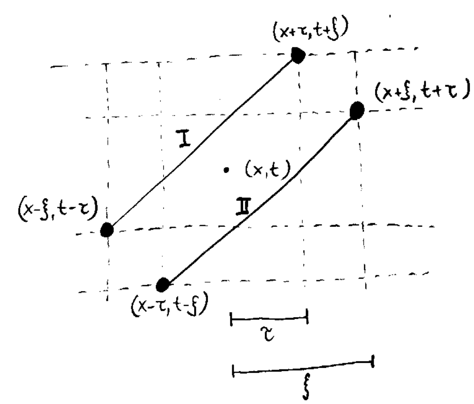
A constructive derivation is given in Evans (and in class).

(b) Set $v = u_t + u_x$. Then, if u satisfies the wave equation,

$$0 = u_{tt} - u_{xx} = v_t - v_x \quad \Rightarrow \quad v = a(x+t)$$

for some function a.

$$\Rightarrow \quad u_t + u_x = a(x+t) \qquad (*)$$



(P)

Lines I and II are both characteristic lines of (*)

To integrate along I, note that, from (*),

$$\frac{d}{ds} u(x-\xi+s,\, t-\tau+s) = a(x-\xi+t-\tau+2s)$$

$$\Rightarrow\ u(x+\tau, t+\xi) - u(x-\xi, t-\tau) = \int_0^{\tau+\xi} a(x-\xi+t-\tau+2s)\, ds$$

Similarly to integrate along II, we note

$$\frac{d}{ds} u(x-\tau+s,\, t-\xi+s) = a(x-\tau+t-\xi+2s)$$

$$\Rightarrow\ u(x+\xi, t+\tau) - u(x-\tau, t-\xi) = \int_0^{\tau+\xi} a(x-\tau+t-\xi+2s)\, ds$$

Since the two integrals above are equal,

$$u(x+\tau, t+\xi) + u(x+\xi, t+\tau) = u(x-\xi, t-\tau) + u(x-\tau, t-\xi). \qquad (**)$$

The converse is proved by differentiating (**) with respect to ξ twice, then setting $\xi=\tau=0$.

<u>Remark:</u> It is known that if u is a solution to the wave equation, then

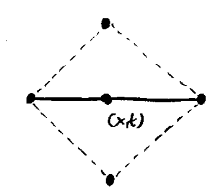
$$u(x,t) = f_1(x+t) + f_2(x-t) \qquad (***)$$

Then (**) can be verified by direct substitution of the respective arguments of u into (***).

The difficulty here lies in the fact that (a) proves (***) only in the absence of boundaries. There are two ways to proceed:

(i) Prove (***) in general by careful integration along characteristics – this will be similar to the direct proof of (**) given above.

(ii) Note that when $\tau=0$, figure (P) is a square



(S)

Thus, the entire square can be seen as the cone of dependence of the initial value problem with initial data on the line $[x-\xi, x+\xi]\times\{t_0=t\}$, both forward and backward in time. So, no boundaries are involved so that (a) directly applies and proves (***).

Now, to prove (**) on a more general polygon (P), tile the polygon with a countable number of squares of type (S) and obtain (**) in the limit.

(c) First obtain u in the closed region A via (a). Due to the finite speed of propagation, this region is not influenced by data outside of $[a,b] \times \{t=0\}$, so that (a) applies.

Then fill B and C using (**), finally D, again via (**) up to $t = b-a$. Now re-initialize to go beyond $t = b-a$.

4. So long as u is a smooth solution, it is determined by the characteristic relation

$$u(x + g(x)t, t) = g(x) \qquad (*)$$

where $\xi(t) = x + g(x)t$ is the characteristic line starting at location x when $t = 0$.

Differentiating (*) with respect to x gives

$$u_x(\xi(t), t)\,(1 + g'(x)t) = g'(x)$$

$$\Rightarrow \quad u_x(\xi(t), t) = \frac{g'(x)}{1 + g'(x)t}$$

Thus, a shock occurs at $(\xi(t), t)$ if $1 + g'(x)t = 0$, so the first shock occurs on the characteristic line originating at location x where g' is most negative. This is a point of inflection.

5. $$\int_U |Du|^2\,dx = \underbrace{\int_{\partial U} u \underset{0}{\underset{\parallel}{n \cdot Du}}\, dS}_{=0} + \underbrace{\int_U u \Delta u\, du}_{\leq \left(\int u^2 dx\right)^{\frac{1}{2}} \left(\int (\Delta u)^2 dx\right)^{\frac{1}{2}}}$$