Real Analysis

Homework 1

due September 8, 2008

1. Let Ω be a set and Σ a σ-algebra on Ω. Show that
   (a) Σ is closed under countable intersection,
   (b) $\Omega \in \Sigma$ and $\emptyset \in \Sigma$,
   (c) $A_1, A_2 \in \Sigma$ implies $A_1 \setminus A_2 \in \Sigma$.

2. The *Borel σ-algebra* on a metric space Ω is defined as the σ-algebra generated by the open subsets of Ω.

   Show that the Borel σ-algebra on $\mathbb{R}$ is also generated by the set of open intervals $\{(a, b) \colon a < b\}$.

   (The proof is essentially given in Folland, the task here is to write it out in a clear, self-contained manner.)

3. (From Folland.) If (Ω, Σ, μ) is a measure space and $A_j \in \Sigma$ for $j \in \mathbb{N}$, show that

$$\mu(\liminf_{j\to\infty} A_j) \leq \liminf_{j\to\infty} \mu(A_j)\,.$$

4. (From Folland.) If (Ω, Σ, μ) is a measure space and $E, F \in \Sigma$, show that

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)\,.$$