Real Analysis

Homework 10

due November 19, 2008

1. (a) Show that for $f \in L^1(\mathbb{R}^n)$,

$$\lim_{y \to 0} \int_{\mathbb{R}^n} |f(x) - f(x-y)| \, dx = 0 \, .$$

   Hint: Use mollifiers.

   (b) Suppose that, moreover, $g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Show that $f * g \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

2. Prove *Parseval's identity*

$$\int_{\mathbb{R}^n} u(x) \, \overline{v}(x) \, \mathrm{d}x = \int_{\mathbb{R}^n} \hat{u}(\xi) \, \overline{\hat{v}}(\xi) \, \mathrm{d}\xi \, .$$

   Note: No need to prove from scratch, you may derive this from the *Plancharel identity*

$$\|u\|_2 = \|\hat{u}\|_2$$

   by a polarization argument.

3. Prove the *Riemann–Lebesgue lemma*, i.e. for $f \in L^1(\mathbb{R}^n)$, $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

   Hint: Note that

$$(f(\,\cdot\, - y))\hat{\ }(\xi) = \mathrm{e}^{-\mathrm{i} y \cdot \xi} \, \hat{f}(\xi) \, .$$

4. Show that the definition of the Fourier transform on L^2 does not depend on the choice of the approximating sequence.