Real Analysis

Homework 8

due November 5, 2008

1. Let (Ω, Σ, μ) be a measure space. Let $f \in L^p(\Omega)$ for $1 \leq p \leq \infty$. Show that for every $\varepsilon > 0$ there exists a $g \in L^\infty(\Omega)$ such that

$$\|f - g\|_p \leq \varepsilon \,.$$

2. Show that the finite unions of half-open intervals for the form $[a, b)$ where $-\infty \leq a \leq b \leq \infty$ and a and b are of the form $n\,2^{-m}$ for some $n \in \mathbb{Z}$, $m \in \mathbb{N}$ form an algebra which generates the Borel σ-algebra on $\mathbb{R}$.

3. (From Lieb and Loss.) Suppose the mollifier $j_\varepsilon \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ and $f \in C(\mathbb{R}^n)$, show that $f_\varepsilon = f * j_\varepsilon$ converges to f pointwise, and it does so uniformly on compact subsets of $\mathbb{R}^n$.

4. Let $f_k(x) = \sin(kx)$. Show that f_k converges weakly, but not strongly to zero in $L^p([0, 1])$ with $1 < p < \infty$.