1. Let (Ω, Σ, μ) be a measure space and $E \in \Sigma$. Show that

$$\mu_E(A) = \mu(A \cap E) \qquad \text{for all } A \in \Sigma$$

defines a measure. (10)

(i) $\mu_E(\emptyset) = \mu(\emptyset) = 0$

(ii) Let $\{A_j\}_{j \in \mathbb{N}} \subset \Sigma$ disjoint. Then

$$\mu_E\left(\bigcup_{j=1}^{\infty} A_j\right) = \mu\left(E \cap \bigcup_{j=1}^{\infty} A_j\right)$$

$$= \mu\left(\bigcup_{j=1}^{\infty} (E \cap A_j)\right)$$

$$= \sum_{j=1}^{\infty} \mu(E \cap A_j) \qquad \text{since } E \cap A_j \text{ disjoint}$$

$$= \sum_{j=1}^{\infty} \mu_E(A) .$$

2. Show that a monotonic function on $\mathbb{R}$ is Borel-measurable. (10)

The level sets of a monotonic function are of the form

(a, ∞) ; $[a, \infty)$; $(-\infty ; a)$; $(-\infty ; a]$; $\mathbb{R}$; $\emptyset$ $(a \in \mathbb{R})$

These are either open or have open complements.

3. Let (Ω, Σ, μ) be a measure space and $f\colon \Omega \to [0,\infty]$ a measurable function. Show that, for $0 < p < \infty$,

$$\int_\Omega f^p \, d\mu = p \int_0^\infty t^{p-1} \mu(\{x \in \Omega : f(x) > t\}) \, dt .$$

(10)

From the definition of the integral:

$$\int_\Omega f^p \, d\mu = \int_0^\infty \mu(\{x \in \Omega : f^p(x) > t\}) \, dt \qquad t = s^p \Rightarrow dt = p s^{p-1} d$$

$$= p \int_0^\infty s^{p-1} \mu(\{x \in \Omega : \underbrace{f^p(x) > s^p}_{\Leftrightarrow f(x) > s}\}) \, ds$$

□

4. *Theorem on differentiation under the integral.*

Let (Ω, Σ, μ) be a measure space and let $I \subset \mathbb{R}$ open. Suppose that $f\colon \Omega \times I \to \mathbb{R}$ satisfies

(i) $f(\cdot, t)$ is measurable for every fixed $t \in I$;

(ii) $f(x, \cdot)$ is differentiable for almost every fixed $x \in \Omega$;

(iii) $|\partial f(x,t)/\partial t| \le g(x)$ for some $g \in L^1(\Omega)$.

Then

$$\frac{d}{dt} \int_\Omega f(x,t) \, d\mu(x) = \int_\Omega \frac{\partial f(x,t)}{\partial t} \, d\mu(x)$$

for every $t \in I$.

(a) Prove the theorem.

(b) Let

$$I(t) = \int_0^\infty \frac{\sin x}{x} e^{-tx} \, dx .$$

Verify that you may differentiate under the integral, then compute $I'(t)$.

(c) Use the result from (b) to prove that

$$\int_0^\infty \frac{\sin x}{x} \, dx = \arctan(\infty) = \frac{\pi}{2} .$$

(10+10+10)

(a) $$\left| \frac{f(x,t+h) - f(x,t)}{h} \right| \le \left| \frac{\partial f(x,t+\tau)}{\partial t} \right| \overset{(iii)}{\le} g(x)$$

for some $\tau = \tau(x,t,h) \in [0,h]$ by the mean value theorem for almost every $x \in \Omega$ and h sufficiently small s.t. $t+h \in I$; the difference quotient is measurable by (i). Then, by dominated convergence,

$$\int_\Omega \lim_{h\to 0} \frac{f(x,t+h) - f(x,t)}{h} \, d\mu = \lim_{h \to 0} \frac{1}{h} \left[\int_\Omega f(x,t+h) \, d\mu - \int_\Omega f(x,t) \, d\mu \right]$$

(b) $$\left|\frac{d}{dt}\left(\frac{\sin x}{x}e^{-tx}\right)\right| = \left|-\sin x\, e^{-tx}\right| \le e^{-tx}$$

For any $t_0>0$, choose $\varepsilon>0$ s.t. $\underbrace{(t_0-\varepsilon, t_0+\varepsilon)}_{=:I} \subset (0,\infty)$

Then (iii) holds with $g(x) = e^{-(t_0-\varepsilon)x}$.

(ii) is clear by the above.

(i) follows from continuity on $(0,\infty)$.

Then

$$I'(t_0) = -\int_0^\infty \sin x\, e^{-t_0x}\,dx = \underbrace{\cos x\, e^{-t_0x}\Big|_0^\infty}_{=-1} + t_0\int_0^\infty \cos x\, e^{-t_0x}\,dx$$

$$= -1 + t_0\Big[\underbrace{\sin x\, e^{-t_0x}\Big|_0^\infty}_{=0} + t_0\underbrace{\int_0^\infty \sin x\, e^{-t_0x}\,dx}_{=-I'(t_0)}\Big]$$

$$\Rightarrow I'(t_0) = \frac{-1}{1+t_0^2} \qquad \forall\, t_0\in(0,\infty).$$

(c) Note that, again by dominated convergence,

$$\lim_{t\to\infty}\int_0^\infty \frac{\sin x}{x}e^{-tx}\,dx = 0$$

(take dominating function $g(x) = e^{-x}$).

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then

$$I(0) = -\int_0^\infty I'(t)\,dt = \int_0^\infty \frac{1}{1+t^2}\,dt = \arctan t\Big|_0^\infty = \frac{\pi}{2}.$$

<u>Note</u>: The integral $\int_0^\infty \frac{\sin x}{x}\,dx$ is understood as the improper integral $\lim_{a\to\infty}\int_0^a \frac{\sin x}{x}\,dx$. Thus, to completely justify the computation, we have to show that

$$\lim_{a\to\infty}\int_0^a f(x)\,dx = \lim_{t\to 0}\int_0^\infty f(x)\,e^{-tx}\,dx$$

for f measurable with $|f|$ summable on any finite $[0,a]$, provided that the left hand limit exists.

<u>Proof</u>: Let $\varepsilon>0$. Since the left hand limit exists, it is Cauchy (w.r.t. any sequence of real numbers $a_n\to\infty$) so that, in particular, $\exists\, a\in\mathbb{R}$ s.t.

$$\sup_{x\ge a}\left|\int_a^x f(y)\,dy\right| < \frac{\varepsilon}{2}.$$

$$\left| \int_0^a f(x)\,dx - \int_0^\infty f(x)\, e^{-tx}\,dx \right|$$

$$\leq \underbrace{\int_0^a |f(x)|\,|1-e^{-tx}|\,dx}_{<\frac{\varepsilon}{2} \text{ for } t \text{ small enough by dom. convergence on } [0,a]} + \underbrace{\left| \int_a^\infty f(x)\, e^{-tx}\,dx \right|}_{\text{integration by parts}}$$

$$= \left| \underbrace{\int_a^x f(y)\,dy\, e^{-tx} \Big|_a^\infty}_{=0} + t \int_a^\infty \int_a^x f(y)\,dy\, e^{-tx}\,dx \right|$$

$$\leq t \int_a^\infty \left| \int_a^x f(y)\,dy \right| e^{-tx}\,dx$$

$$\leq \underbrace{\sup_{x \geq a} \left| \int_a^x f(y)\,dy \right|}_{<\frac{\varepsilon}{2}} \quad \underbrace{t \int_a^\infty e^{-tx}\,dx}_{= -e^{-tx}\big|_a^\infty = e^{-ta} \leq 1}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

5. Let (Ω, Σ, μ) be a finite measure space.
Show that for $1 \leq p \leq r \leq \infty$, $L^p(\Omega) \supset L^r(\Omega)$. (10)

Let $f \in L^r$

Hölder

$$\Rightarrow \int_\Omega |f|^p\,d\mu \leq \|1\|_q \, \|f^p\|_{q'} \quad , \quad \frac{1}{q} + \frac{1}{q'} = 1$$

where we require that $q'p = r$

$\Rightarrow q' = \frac{r}{p} \geq 1$

$\Rightarrow q \in [1, \infty]$

Since $\mu(\Omega) < \infty$, $\|1\|_q < \infty$

and $\|f^p\|_{q'} = \|f\|_r^p < \infty$,

the LHS of the inequality is finite, too.

6. Let (Ω, Σ, μ) be a measure space, $1 < p \le \infty$ and $1/p + 1/q = 1$. Let $\ell \in L^p(\Omega)^*$ be represented by a function $g \in L^q(\Omega)$ in the sense that

$$\ell(f) = \int_\Omega f\,g\,d\mu$$

for every $f \in L^p(\Omega)$. Show that

$$\|\ell\|_{L^p(\Omega)^*} = \|g\|_{L^q(\Omega)}.$$

(10)

$$\|\ell\|_{L^p(\Omega)^*} = \sup_{\|f\|_p=1} |\ell(f)|$$

$$= \sup_{\|f\|_p=1} \left| \int_\Omega f g \, d\mu \right| \le \sup_{\|f\|_p=1} \|f\|_p \|g\|_q = \|g\|_q$$

by Hölder's inequality.

Vice versa,

$$\|\ell\|_{L^p(\Omega)^*} \ge |\ell(f)| \qquad \text{for every } f \in L^p \text{ with } \|f\|_p$$

For $1<p\le\infty$, choose $f = \begin{cases} \bar{g}\,|g|^{q-2}\,\|g\|_q^{q-1} & \text{if } g(x) \neq 0 \\ 0 & \text{if } g(x) = 0 \end{cases}$

$$\Rightarrow \ell(f) = \int |g|^q \, d\mu \, \|g\|_q^{q-1} = \|g\|_q$$

Note that WLOG we can assume $\ell \neq 0$ s.t. $\|g\|_q \neq 0$,

and, by direct calculation, that $f \in L^p$.

7. (a) Find a sequence of bounded, Lebesgue-measurable sets in $\mathbb{R}$ whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to a function f with the property that 2f is a characteristic function of a set with positive measure.

Note: Full credit if you verify that your example has the requested properties on some (nontrivial) subspace of L^2.

(b) How about the possibility that f/2 is a characteristic function?

(10+10)

(a) Let $x_i^n = h_n i$ with $h_n = \frac{1}{n}$, $i = 0, \dots, n$

$I_i^n = [x_{i-1}, x_i)$ $\qquad i = 1, \dots, n$

$$f_n = \sum_{\substack{i=1 \\ i \text{ odd}}}^{n} \chi_{I_i^n}$$

<u>Claim</u>: $f_n \rightharpoonup \frac{1}{2}\chi_{[0,1]}$ in L^2

for simplicity, consider only the subsequence with n even

Let $g = \sum_{i=-\infty}^{\infty} a_i \chi_{I_i^N} \in L^2(\mathbb{R})$

Then $\int_{\mathbb{R}} f_n g \, dx = \frac{1}{2}\sum_{i=1}^{N} a_i h_N = \int_{\mathbb{R}} \frac{1}{2}\chi_{[0,1]}\, g$

$\forall\, n > N$, n even

It is known (not to be proved here, the argument is essentially given by Lieb & Loss, 1.17) that such functions are dense in L^2, hence the general statement on weak-L^2- convergence.

(b) Suppose $f = 2\chi_A$ for some A
(WLOG $\mu(A) < \infty$, if not, intersect everything with set of finite measure)

Then, since $f_j \to f$,

$$\int f_j \chi_A \longrightarrow \int f \chi_A = 2\mu(A)$$

On the other hand,

$$\int f_j \chi_A \leq \mu(A) \qquad \text{since } f_j \leq 1 \text{ pointwise}$$

$\Longrightarrow$ contradiction!