# Partial Differential Equations

Homework 5

due December 7, 2009

1. Evans, p. 426, Problem 6

2. Assume $W \subset \mathbb{R}^n$ is open, $u\colon \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is a smooth time-dependent vector field with flow $\phi_t$, i.e. $\partial_t \phi_t(a) = u(\phi_t(a), t)$, and $W_t = \phi_t(W)$.

   Use Liouville's theorem to prove that

   (a) $\displaystyle \frac{d}{dt} \int_{W_t} \rho \, f \, dx = \int_{W_t} \rho \, (\partial_t + u \cdot \nabla) f \, dx$, where $\partial_t \rho + \nabla \cdot (\rho u) = 0$, and

   (b) $\displaystyle \frac{d}{dt} \int_{W_t} f \, dx = \int_{W_t} \partial_t f \, dx + \int_{\partial W_t} f \, \nu \cdot u \, dS.$

3. Recall the construction of the function spaces for solutions of the Navier–Stokes equations, where for $U \in \mathbb{R}^n$ open, bounded, with $C^2$ boundary,

$$\mathcal{V} = \{ u \in C_c^\infty(U, \mathbb{R}^n) \colon \nabla \cdot u = 0 \} \,,$$

   $H$ is the closure of $\mathcal{V}$ in $L^2$, and $V$ is the closure of $\mathcal{V}$ in $H^1$. Recall the continuous trace operator $\gamma\colon E(U) \to H^{-1/2}(\partial U)$ which satisfies $\gamma(u) = \nu \cdot u$ if $u \in C^\infty(\overline{U})$, where

$$E(U) = \{ u \in L^2(U, \mathbb{R}^n) \colon \nabla \cdot u \in L^2(U, \mathbb{R}) \}$$

   with the divergence understood in the sense of weak derivatives.

   Prove that

$$H = \{ u \in L^2(U, \mathbb{R}^n) \colon \nabla \cdot u = 0 \text{ and } \gamma(u) = 0 \} \,.$$

4. In the notation of the previous question, let $P$ denote the $L^2$-orthogonal projector onto $H$ and define the *Stokes operator* $A\colon \mathcal{D}(A) \to H$ by $A = -P\Delta$ with domain $\mathcal{D}(A) = H^2(U, \mathbb{R}^n) \cap V$.

   Recall from class that if $f \in H$, then $A^{-1} f = u$ is the unique solution in $\mathcal{D}(A)$ of the Stokes equations. (This was proved by an application of the a Lax–Milgram theorem followed by an elliptic regularity argument.)

(a) Show that

$$\langle Au, v\rangle_{L^2(U,\mathbb{R}^n)} = \langle u, v\rangle_{H^1(U,\mathbb{R}^n)}$$

for all $u, v \in \mathcal{D}(A)$, thus conclude that $A$ is a symmetric operator.

*Note:* First assume $u, v \in \mathcal{V}$, then argue by density. Note also that

$$\langle u, v\rangle_{H^1(U,\mathbb{R}^n)} = \int_U \nabla u : \nabla v \, dx \, .$$

(b) Show that $A$ is self-adjoint.

*Note:* By part (a), $A$ is symmetric, so it remains to be shown that $\mathcal{D}(A) = \mathcal{D}(A^*)$, where

$$\mathcal{D}(A^*) = \{u \in H \colon v \mapsto \langle u, Av\rangle_{L^2} \text{ is bounded}\} \, .$$

(I.e., $v \mapsto \langle u, Av\rangle_{L^2}$ can be represented by $\langle f, v\rangle_{L^2}$ by the Riesz representation theorem.)

(c) Show that $A^{-1}$ is a compact operator on $H$.

5. Due to the results in the previous question, there exists a complete $L^2$-orthonormal sequence $\{w_j\}$ of eigenfuctions of $A$ with corresponding eigenvalues $\lambda_j$, where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ and $w_j \in \mathcal{D}(A)$ for $j \in \mathbb{N}$.

   Thus, if

$$u = \sum_{j=1}^{\infty} u_j \, w_j \, ,$$

   we can define $A^\alpha$ for any $\alpha \in \mathbb{R}$ by

$$A^\alpha u = \sum_{j=1}^{\infty} \lambda_j^\alpha \, u_j \, w_j \, ,$$

$$\langle u, v\rangle_{\mathcal{D}(A^\alpha)} = \sum_{j=1}^{\infty} \lambda^{2\alpha} \, u_j \, v_j \quad \text{when } u = \sum_{j=1}^{\infty} u_j \, w_j \text{ and } v = \sum_{j=1}^{\infty} u_j \, v_j$$

   with domain

$$\mathcal{D}(A^\alpha) = \{u \in H \colon \|u\|^2_{\mathcal{D}(A^\alpha)} = \langle u, u\rangle_{\mathcal{D}(A^\alpha)} < \infty\} \, .$$

   Show that $D(A^{1/2}) = V$.

6. In the notation above, let $u \in V$. Prove that, for $n \leq 3$, there exists a constant $c$ such that

$$\|u \cdot \nabla u\|_{V^*} \leq c \, \|u\|_V^{\frac{n}{2}} \, \|u\|_H^{2-\frac{n}{2}} \, .$$

   *Note:* Due to the result in the previous question, we can identify $V^* = \mathcal{D}(A^{-1/2})$, so that

$$\|v\|_{V^*} = \|A^{-1/2} v\|_{L^2} \quad \text{and} \quad \langle v, w\rangle_{V^*,V} = \langle A^{-1/2} v, A^{1/2} w\rangle_{L^2} \, .$$

You will also need the Sobolev inequality in the form given as equation (14) in the proof of Theorem 5.6.1 in Evans (you can also quote the result from other sources where the exponents are more explicit).

7. In the notation above, suppose that

$$\begin{aligned} u_m &\rightharpoonup 0 \qquad \text{weakly in } L^2((0,T);V)\,, \\ u_m' &\rightharpoonup 0 \qquad \text{weakly in } L^p((0,T);V^*)\,. \end{aligned}$$

for some $p > 1$.

(a) Prove that

$$u_m \to 0 \qquad \text{in } C([0,T];V^*)\,.$$

(b) Then conclude that

$$u_m \to 0 \qquad \text{in } L^2((0,T);H)\,.$$

*Hint:* For part (a), use the fundamental theorem of calculus, estimate in the $V^*$-norm, and employ an argument similar to the one given in part (1) of the proof of Evans, Theorem 5.9.4 on p. 287.

You will also need to refer to the compactness of the embedding $V \subset H$.