Partial Differential Equations

Final Exam

December 15, 2009

Last Name: ______________________

First Name: ______________________

Signature: ______________________

In this exam, you will prove existence, uniqueness, and some regularity for weak solutions of the nonlinear parabolic PDE. The procedure is broken up to 10 parts of 5 points each.

Let $U \subset \mathbb{R}^n$ be open, bounded, and with C^2 boundary, and set $U_T = U \times (0, T]$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be C^1 and suppose that there exist constants c_1 and c_2 such that

$$|f(z)| \le c_1 |z| \qquad \text{and} \qquad |f'(z)| \le c_2$$

for all $z \in \mathbb{R}$, and let $g \in L^2(U)$.

Under these conditions, consider the initial-boundary value problem for $u\colon \overline{U}_T \to \mathbb{R}$,

$$\left.\begin{aligned} u' - \Delta u &= f(u) && \text{in } U_T, \\ u &= 0 && \text{on } \partial U \times [0, T], \\ u &= g && \text{on } U \times \{t = 0\}, \end{aligned}\right\} \qquad (*)$$

where $u' = \partial_t u$.

1. Define a sensible notion of weak solution for this problem.

   Note: You might first want to revisit this notion after solving some of the following questions.

Suppose $u \in C([0,T]; L^2) \cap L^2((0,T); H_0^1)$

$u' \in L^2((0,T); H^{-1})$,

then we say u satisfies $(*)$ weakly if $u(0) = g$ and

$$\langle u', v \rangle_{H^{-1}, H_0^1} + \langle Du, Dv \rangle_{L^2} = \langle f(u), v \rangle_{L^2} \qquad \forall v \in H_0^1$$

a.e. on $t \in [0,T]$.

Note that when u is smooth, a weak solution is a strong solution as is easily seen by integration by parts on the second term.

2. Define a sequence of approximate solutions u_m by Galerkin truncation.

Note: What basis do you choose? Give a careful argument as to why the basis of your choice exists. What can you say about the existence of these approximate solutions?

Let $L = -\Delta$ with homogeneous Dirichlet boundary conditions.

By a standard application of the Lax-Milgram Theorem, $L: H_0^1 \to H^{-1}$ is invertible; since H_0^1 is compactly embedded in L^2, L^{-1} is a compact operator on L^2. Hence, L^{-1} and thus L have a complete set of orthonormal eigenfunctions (w.r.t. the L^2 inner product) $\{w_j\}_{j=1}^{\infty}$, where $w_j \in H_0^1$ for all $j \in \mathbb{N}$. Let this set be ordered by magnitude of the eigenvalues and let P_m be the L^2-orthogonal projector onto $\text{span}\{w_1, \dots, w_m\}$.

We define the approximate equation

$$U_m' - \Delta U_m = P_m f(U_m)$$
$$u_m(0) = P_m g$$

which is a Lipschitz system of ODEs, thus has a solution on some interval $[0, T_m)$ by Picard-Lindelöf. Moreover, if T_m is maximal,

then $\|U_m(t)\| \to \infty$ as $t \to T_m$ for any norm.

3. **Show that $\{u_m\}$ is bounded in $L^\infty((0,T);L^2(U)) \cap L^2((0,T);H^1_0(U))$, and that $\{u'_m\}$ is bounded in $L^2((0,T);H^{-1}(U))$.**

Integrate the approximate equation against u_m:

$$**)\quad (u_m', u_m)_{L^2} + (Du_m, Du_m)_{L^2} = (u_m, f(u_m))_{L^2}$$

As $u_m \in \langle w_1, \dots, w_m \rangle \in H^1_0(U)$ by Poincaré's inequality and due to $(u_m', u_m)_{L^2} = \frac{1}{2}\frac{d}{dt}\left(\|u_m\|^2_{L^2}\right)$ for a.e. $t \in [0,T]$

$$*)\quad \frac{d}{dt}\left(\|u_m\|^2_{L^2}\right) + 2\beta \|u_m\|^2_{H^1_0} \leq \frac{1}{2}\|u_m\|^2_{L^2} + \frac{1}{2}\|f(u_m)\|^2_{L^2}$$
$$\leq \frac{1}{2}\|u_m\|^2_{L^2} + \frac{1}{2}c_1^2\|u_m\|^2_{L^2} = \frac{1}{2}(1+c_1^2)\|u_m\|^2_{L^2}$$

(i) From Gronwall's inequality with $y(t) := \|u_m(t)\|^2_{L^2}$

from $y'(t) \leq \frac{1}{2}(1+c_1^2)\, y(t)$

we get

$$\|u_m(t)\|^2_{L^2} \leq e^{\frac{1}{2}(1+c_1^2)t} \|u_m(0)\|^2_{L^2} \leq e^{\frac{1}{2}(1+c_1^2)t} \|g\|^2_{L^2} \qquad t\in[0,T]$$

(with $u_m(0) = P_m g$ used in the second inequality)

in particular

$$\max_{0\leq t\leq T} \|u_m(t)\|^2_{L^2} \leq e^{\frac{1}{2}(1+c_1^2)T} \|g\|^2_{L^2}$$

(ii) From $(*)$

$$2\beta \int_0^T \|u_m\|^2_{H^1_0}\, dt \leq \frac{1}{2}(1+c_1^2) \int_0^T \|u_m\|^2_{L^2}\, dt \overset{(i)}{\leq} C(T) \|g\|^2_{L^2}$$

(iii) From the weak form of the projected equation $u_m' + \Delta u_m = P_m f(u_m)$, i.e. from $(**)$, we deduce

$$\langle u_m', v\rangle_{H^{-1}, H^1_0} = (u_m', v)_{L^2} = (u_m', v^1)_{L^2} = (f(u_m), v^1)_{L^2} + (Du_m, Dv^1)$$

(the first equality by $H^1_0 \subset L^2 \cong (L^2)' \subset H^{-1}$)

Thus, by the orthogonality of $\{w_k\}_{k\in\mathbb{N}}$ in $H^1_0(U)$

$$\|u_m'\|_{H^{-1}} = \sup_{\|v\|_{H^1_0}\leq 1} |\langle u_m', v\rangle_{H^{-1}, H^1_0}| \leq C \|u_m\|_{H^1_0}$$

and consequently

$$\int_0^T \|u_m'\|^2_{H^{-1}}\, dt \overset{(ii)}{\leq} C(T) \|g\|^2_{L^2}$$

4. Conclude the existence of subsequences, still denoted u_m and u'_m, such that

$$\begin{aligned} u_m &\rightharpoonup u \qquad \text{weakly in } L^2((0,T); H_0^1(U)), \\ u'_m &\rightharpoonup v \qquad \text{weakly in } L^2((0,T); H^{-1}(U)) \end{aligned}$$

and show that $u' = v$.

4. $L^2((0,T);H_0^1(U))$ and $L^2((0,T);H^{-1}(U))$ are reflexive Banach spaces and by problem 3.

$$\|u_m\|_{L^2((0,T);H_0^1)} \leq C$$

$$\|u_m'\|_{L^2((0,T);H^{-1})} \leq C$$

where C is independent of m.

Then there exists subsequences, still denoted u_m, u_m' such that

$$u_m \rightharpoonup u \quad \text{weakly in } L^2((0,T);H_0^1)$$
$$u_m' \rightharpoonup v \quad \text{weakly in } L^2((0,T);H^{-1})$$

$v \in L^1((0,T);H^{-1})$ is weak derivative of $u \in L^1((0,T);H^{-1})$ provided

$$(*) \qquad \int_0^T \phi'(t)\, u(t)\, dt = - \int_0^T \phi(t)\, v(t)\, dt \qquad \forall \phi \in C_c^\infty(0,T)$$

We have the following statement in L^2 for arbitrary $\phi \in C_c^\infty(0,T)$:

$$\int_0^T \phi'(t)\, u_m(t)\, dt = -\int_0^T \phi(t)\, u_m'(t)\, dt$$

testing against arbitrary $w \in H_0^1 \subset L^2$ and applying Theorem 8 in Appendix E.5 in Evans gives:

$$\left(w, \int_0^T \phi'(t)\, u_m(t)\, dt\right)_{L^2} = \left(w, -\int_0^T \phi(t)\, u_m'(t)\, dt\right)_{L^2}$$

$$\int_0^T (w, \phi'(t)\, u_m(t))_{L^2}\, dt \qquad - \int_0^T (w, \phi(t)\, u_m'(t))_{L^2}\, dt$$

$$\int_0^T \langle w\phi'(t), u_m(t)\rangle_{H^{-1},H_0^1}\, dt \qquad - \int_0^T \langle u_m'(t), w\phi(t)\rangle_{H^{-1},H_0^1}\, dt$$

$$\downarrow m\to\infty \qquad\qquad \downarrow m\to\infty$$

$$\int_0^T \langle w\phi'(t), u(t)\rangle_{H^{-1},H_0^1}\, dt \qquad - \int_0^T \langle v(t), w\phi(t)\rangle_{H^{-1},H_0^1}\, dt$$

$$\int_0^T \langle w, \phi'(t)\, u(t)\rangle_{H_0^1,H^{-1}}\, dt \qquad - \int_0^T \langle w, \phi(t)\, v(t)\rangle_{H_0^1,H^{-1}}\, dt$$

$$\Longleftrightarrow \left\langle w, \int_0^T \phi'(t)\, u(t)\, dt\right\rangle_{H_0^1,H^{-1}} = \left\langle w, -\int_0^T \phi(t)\, v(t)\, dt\right\rangle_{H_0^1,H^{-1}} \quad \forall w \in H_0^1$$

$\Longrightarrow$ $(*)$ $\square$

5. Prove that the limit function u indeed satisfies the weak formulation of the problem from problem (1).

   Note: The attainment of the initial condition will be considered later.

5. For $m \geq N$ take an arbitrary test fct

$$v_N := \sum_{k=1}^{N} d^k w_k \quad \in \langle w_1, \dots, w_N \rangle \subset \langle w_1, \dots, w_m \rangle$$

Integrating with respect to t the projection of the non linear problem

$$\int_0^T \langle u_m', v_N \rangle_{H^{-1}, H_0^1} \, dt + \int_0^T (Du_m, Dv_N)_{L^2} \, dt = \int_0^T (f(u_m), v_N)_{L^2} \, dt$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\int_0^T \langle u', v_N \rangle_{H^{-1}, H_0^1} \, dt + \int_0^T (Du, Dv_N)_{L^2} \, dt = \int_0^T (f(u), v_N)_{L^2} \, dt$$

$\forall v_N$ as given above

By the weak convergences from problem 4 and the mean value theorem,

$$\left| \int_0^T (f(u_m(t)) - f(u(t)), v_N)_{L^2} \, dt \right| = \left| \int_0^T (f'(\xi_t)(u_m(t) - u(t)), v_N)_{L^2} \, dt \right|$$

$$= \underbrace{|f'(\xi_t)|}_{\leq C_2} \left| \int_0^T (u_m(t) - u(t), v_N)_{L^2} \, dt \right| \xrightarrow{m \to \infty} 0$$

as $u_m \longrightarrow u$ in $L^2((0,T); H_0^1(U))$

As $\left\{ \frac{w_k}{\sqrt{\lambda_k}} \right\}_{k=1,\dots,\infty}$ is an orthonormal basis of $H_0^1(U)$, $\{ v_N : N \in \mathbb{N} \}$ is a dense subset of H_0^1 and in fact

$$\int_0^T \langle u', v \rangle_{H^{-1}, H_0^1} \, dt + \int_0^T (Du, Dv)_{L^2} \, dt = \int_0^T (f(u), v)_{L^2} \, dt \qquad \forall v \in H_0^1(U),$$

in particular

$$\langle u', v \rangle_{H^{-1}, H_0^1} + (Du, Dv)_{L^2} = (f(u), v)_{L^2} \qquad \forall v \in H_0^1(U) \quad \text{a.e. } 0 \leq t \leq T.$$

6. Show that $u \in C([0,T]; L^2(U))$.

Note: Explicit proof is required. However, there is no need for using mollifiers; you can use the approximating sequence u_m you already have.

define the regularizations $u^\varepsilon := \eta_\varepsilon * u$ of $u \in L^2((0,T); H^1_0(U))$, $u' \in L^2((0,T); H^{-1}(U))$,

then $\frac{d}{dt} \|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2} = 2\left(u^{\varepsilon\prime}(t) - u^{\delta\prime}(t), u^\varepsilon(t) - u^\delta(t)\right)_{L^2}$

$\overset{\int_s^t}{\Rightarrow} \quad \|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2} = \|u^\varepsilon(s) - u^\delta(s)\|^2_{L^2} + 2\int_s^t \langle u^{\varepsilon\prime}(\tau) - u^{\delta\prime}(\tau), u^\varepsilon(\tau) - u^\delta(\tau)\rangle d\tau$

for all $0 \le s, t \le T$.

As $u^\varepsilon \to u$ in $L^2((0,T); L^2(U))$, there exists a subsequence still denoted u^ε such that $u^\varepsilon(s) \xrightarrow{L^2} u(s)$ a.e. $s \in [0,T]$. Fix a point $s \in (0,T)$ such that $u^\varepsilon(s) \to u(s)$ in $L^2(U)$.

Then

$$\limsup_{\varepsilon,\delta \to 0} \max_{0 \le t \le T} \|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2} \le \limsup_{\varepsilon,\delta \to 0} \int_0^T \left(\|u^{\varepsilon\prime}(\tau) - u^{\delta\prime}(\tau)\|^2_{H^{-1}} + \|u^\varepsilon(\tau) - u^\delta(\tau)\|^2_{H^1_0} \right) d\tau = 0$$

As $\left(C([0,T]; L^2(U)), \|\cdot\|_{C([0,T]; L^2(U))} \right)$ is a complete normed space, the Cauchy-sequence u^ε has a unique limit $v \in C([0,T]; L^2(U))$, with $u = v$ a.e. $t \in [0,T]$ as $u^\varepsilon(t) \to u(t)$ a.e. t, i.e. u has a continuous representative in $C([0,T]; L^2(U))$.

Alternatively, use the sequence u_m instead of u^ε in the above proof!

7. Show that $t \mapsto \|u(t)\|_{L^2}^2$ is absolutely continuous with

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = \langle u', u\rangle_{H^{-1},H_0^1}$$

for a.e. $t \in [0,T]$.

As in problem 6, the mollification u^ε satisfies

$$\|u^\varepsilon(t)\|_{L^2}^2 = \|u^\varepsilon(s)\|_{L^2}^2 + 2\int_s^t \langle u^{\varepsilon\prime}(\tau), u^\varepsilon(\tau)\rangle_{H^{-1},H_0^1}\, d\tau$$

and upon passing to the limit

$$\|u(t)\|_{L^2}^2 = \|u(s)\|_{L^2}^2 + 2\int_s^t \langle u'(\tau), u(\tau)\rangle\, d\tau \qquad \forall\ 0 \le s, t \le T.$$

Consequently,

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = 2\langle u'(t), u(t)\rangle_{H^{-1},H_0^1} \qquad \text{for a.e. } 0 \le t \le T.$$

8. Show that $u(0) = g$.

As in the proof of problem 5, by choosing time-dependent test fcts $v_N(t) := \sum_{k=1}^{N} d^k(t)\, w_k \in C^1([0,T]; H_0^1(U))$ we can derive

$$\int_0^T \langle u', v_N\rangle_{H^{-1},H_0^1}\, dt + \int_0^T (Du, Dv_N)_{L^2}\, dt = \int_0^T (f(u), v_N)_{L^2}\, dt$$

and by density

$$\int_0^T \langle u', v\rangle\, dt + \int_0^T (Du, Dv)_{L^2}\, dt = \int_0^T (f(u), v)_{L^2}\, dt \quad \forall\, v \in L^2((0,T); H_0^1(U)).$$

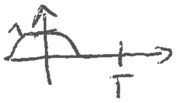
upon choosing a test fct $v \in C^1([0,T]; H_0^1(U))$ with $v(T) = 0$ it follows

$$-\int_0^T \langle v', u\rangle + (Du, Dv)\, dt = \int_0^T (f(u), v)\, dt + (u(0), v(0))$$

analogously,

$$-\int_0^T \langle v', u\rangle + (Du, Dv)\, dt = \int_0^T (f(u), v)\, dt + (g, v(0))$$

$$\Rightarrow \quad (u(0) - g, v(0)) = 0 \quad \text{for arbitrary } v \in C^1([0,T]; H_0^1(U))$$

Therefore $u(0) = g$ [choose, e.g., $v(t) = \lambda(t) w_k$, $k = 1, 2, \dots$ with $\lambda \in C_c^\infty(\mathbb{R})$ ]

9. Prove uniqueness of weak solutions.

Let u_1, u_2 be two solutions, then $u := u_1 - u_2$ satisfies

$$\begin{cases} u' - \Delta u = f(u_1) - f(u_2) & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = 0 & \text{on } U \times \{t=0\} \end{cases}$$

By the mean value theorem $\exists\, \xi_t \in (u_1(t), u_2(t))$ such that

$$f(u_1) - f(u_2) = f'(\xi_t)\,(u_1 - u_2)$$

$$\begin{cases} u' - \Delta u - f'(\xi_t)\, u = 0 & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = 0 & \text{on } U \times \{0\} \end{cases}$$

observe that $c(x,t) := f'(\xi_t) \in L^\infty(U_T)$ as by hypothesis $|f'(z)| \leq c_2$.

So, we are in the general framework of a uniformly parabolic PDE, for which there exists a unique solution $u \equiv 0$.

10. Show that if $g \in H_0^1(U)$, then

$$u \in L^\infty((0,T); H_0^1(U)) \cap L^2((0,T); H^2(U)).$$

The weak form for the projected equation reads

$$(u_m', v)_{L^2} + (Du_m, Dv)_{L^2} = (f(u_m), v)_{L^2} \quad \forall v \in \langle w_1, \dots, w_m \rangle$$

$$\Rightarrow (u_m', u_m') + (Du_m, Du_m') = (f(u_m), u_m')$$

$$\Rightarrow \|u_m'\|_{L^2}^2 + \frac{d}{dt}\left(\frac{1}{2}\|Du_m\|_{L^2}^2\right) = (f(u_m), u_m')_{L^2} \leq \frac{C}{\varepsilon}\|f(u_m)\|_{L^2}^2 + \varepsilon\|u_m'\|_{L^2}^2$$

$$\leq \frac{C c_1^2}{\varepsilon}\|u_m\|_{L^2}^2 + \varepsilon\|u_m'\|_{L^2}^2$$

Choosing $\varepsilon = \frac{1}{2}$ and integrating $\int_0^t \dots dt$, taking $\sup_{0 \leq t \leq T} \dots$, we find

$$\int_0^T \|u_m'\|_{L^2}^2 \, dt + \sup_{0 \leq t \leq T} \|Du_m(t)\|_{L^2}^2 \leq C\left(\|Du_m(0)\|_{L^2}^2 + \int_0^T \|u_m\|_{L^2}^2 \, dt\right)$$

(+)

$$\leq C\left(\|u_m(0)\|_{H_0^1}^2 + C(T)\|g\|_{L^2}^2\right)$$

$$\leq C\left(\|g\|_{H_0^1}^2\right)$$