Engineering and Science Mathematics 1A

Final Exam

December 13, 2010

Some trigonometric identities:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sin 2x = 2\,\sin x\,\cos x$$
$$1 + \tan^2\theta = \sec^2\theta \qquad \cos 2x = 2\,\cos^2 x - 1 = 1 - 2\,\sin^2 x$$
$$1 + \cot^2\theta = \csc^2\theta \qquad \tan 2x = \frac{2\,\tan x}{1 - \tan^2 x}$$
$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Useful integrals:

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C$$
$$\int \frac{du}{1+u^2} = \arctan u + C = -\operatorname{arccot} u + C'$$
$$\int \frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec}|u| + C$$
$$\int \sec u \, du = \ln\left|\sec u + \tan u\right| + C$$

1. Compute the limits

   (a) $\lim\limits_{n\to\infty} \dfrac{\sin n}{n}$

   (b) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$

   (c) $\lim\limits_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}$

(5+5+5)

2. Prove that the equation

$$x^3 - 4x + 1 = 0$$

has three (real) solutions.

Hint: Don't try to compute the solutions! (10)

3. Consider the function
$$f(x) = \frac{\ln x}{x}.$$
What is the domain of f? Find horizontal and vertical asymptotes, local minima, local maxima, and inflection points of f. Identify the regions where the graph of f is concave upward or concave downward. Finally, sketch the graph into the coordinate system provided. (15)

4. Compute the indefinite integrals

   (a) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

   (b) $\displaystyle\int \frac{4x-2}{x^3-x}\,dx$

(10+10)

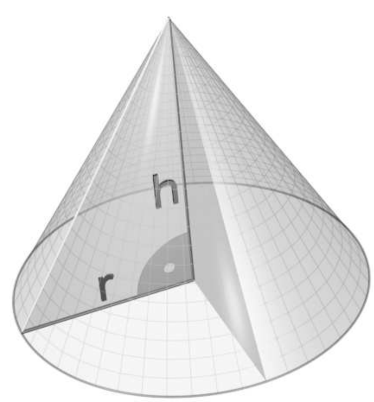
5. Compute the improper integrals

   (a) $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

   (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{(\cos x)^{4/3}}\,dx$

(10+10)

6. Show, using integration, that the volume of a right circular cone of height h and radius r (see Figure) is given by (10)
$$V = \frac{1}{3}\pi h r^2.$$



7. Compute the Taylor series about the point $x = 0$ of

   (a) $f(x) = \dfrac{1}{1-x}$

   (b) $f(x) = \dfrac{1-x^2}{1-x}$

(c) $f(x) = \dfrac{1-x^3}{1-x}$

(d) Do you see a pattern? Can you formulate a more general statement?

(5+5+5+5)

8. For which values of p does the series

$$\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^p}$$

converge, respectively diverge? (10)

9. Let $P(t)$ denote the number of fish in a lake at time t, and let C denote the "carrying capacity" of the lake. Suppose further that fishermen catch a fraction k of the fish per unit of time, so that the population satisfies the equation

$$\frac{dP}{dt} = \left(1 - \frac{P}{C}\right) P - k\,P \qquad \text{with} \qquad P(0) = P_0\,.$$

(a) For given values of C and k, when is the population increasing, when is it decreasing?

(b) For given values of C and k, how many fish will be in the lake in the long run? (You do not need to solve the differential equation to answer this question!)

(c) Which value of the fishing rate k maximizes the number of fish caught in the long run?

(d) Solve the differential equation explicitly and check that your solution is consistent with your answer to part (b).

(5+5+10+10)