about 3 h 25 min. So we see that—whereas a natural (but naive and ***wrong***) guess might have been 2:00 P.M. (one more hour for the remaining 5 ft of water to drain)—the tank actually is not empty until about 3:25 P.M. You should use Eq. (25) to show that the actual water depth in the tank at 2:00 P.M. is about 1.72 ft, and the depth at 3:00 P.M. is about 2 in., so it takes about 25 min for the last 2 in. of water in the tank to drain!

EXAMPLE 5 A hemispherical tank has top radius 4 ft and, at time $t = 0$, is full of water. At that moment a circular hole of diameter 1 in. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?

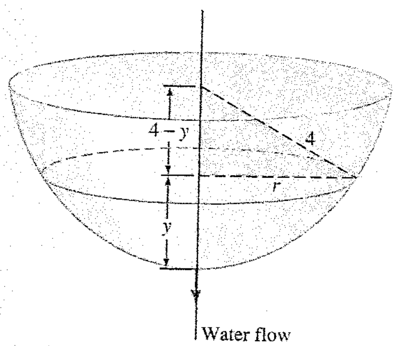

Fig. 6.5.5 Draining a hemispherical tank

Solution From the right triangle in Fig. 6.5.5, we see that

$$A(y) = \pi r^2 = \pi[16 - (4 - y)^2] = \pi(8y - y^2).$$

With $g = 32\ \text{ft/s}^2$, Eq. (20) takes the form

$$\pi(8y - y^2)\frac{dy}{dt} = -\pi\left(\frac{1}{24}\right)^2\sqrt{64y};$$

$$\int (8y^{1/2} - y^{3/2})\,dy = -\int \frac{1}{72}\,dt + C;$$

$$\frac{16}{3}y^{3/2} - \frac{2}{5}y^{5/2} = -\frac{1}{72}t + C.$$

Now $y(0) = 4$, so

$$C = \tfrac{16}{3}\cdot 4^{3/2} - \tfrac{2}{5}\cdot 4^{5/2} = \tfrac{448}{15}.$$

The tank is empty when $y = 0$—that is, when

$$t = 72 \cdot \tfrac{448}{15} \approx 2150 \quad \text{(s)},$$

about 35 min 50 s. So it takes slightly less than 36 min for the tank to drain.

6.5 PROBLEMS

Find general solutions (implicit if necessary, explicit if possible) of the differential equations in Problems 1 through 10.

1. $\dfrac{dy}{dx} = 2x\sqrt{y}$

2. $\dfrac{dy}{dx} = 2xy^2$

3. $\dfrac{dy}{dx} = x^2y^2$

4. $\dfrac{dy}{dx} = (xy)^{3/2}$

5. $\dfrac{dy}{dx} = 2x\sqrt{y - 1}$

6. $\dfrac{dy}{dx} = 4x^3(y - 4)^2$

7. $\dfrac{dy}{dx} = \dfrac{1 + \sqrt{x}}{1 + \sqrt{y}}$

8. $\dfrac{dy}{dx} = \dfrac{x + x^3}{y + y^3}$

9. $\dfrac{dy}{dx} = \dfrac{x^2 + 1}{x^2(3y^2 + 1)}$

10. $\dfrac{dy}{dx} = \dfrac{(x^3 - 1)y^3}{x^2(2y^3 - 3)}$

Solve the initial value problems in Problems 11 through 20.

11. $\dfrac{dy}{dx} = y^2,\ y(0) = 1$

12. $\dfrac{dy}{dx} = \sqrt{y},\ y(0) = 4$

13. $\dfrac{dy}{dx} = \dfrac{1}{4y^3},\ y(0) = 1$

14. $\dfrac{dy}{dx} = \dfrac{1}{x^2y},\ y(1) = 2$

15. $\dfrac{dy}{dx} = \sqrt{xy^3},\ y(0) = 4$

16. $\dfrac{dy}{dx} = \dfrac{x}{y},\ y(3) = 5$

17. $\dfrac{dy}{dx} = -\dfrac{x}{y},\ y(12) = -5$

18. $y^2\dfrac{dy}{dx} = x^2 + 2x + 1,\ y(1) = 2$

19. $\dfrac{dy}{dx} = 3x^2y^2 - y^2,\ y(0) = 1$

20. $\dfrac{dy}{dx} = 2xy^3(2x^2 + 1),\ y(1) = 1$

21. Suppose that the fish population $P(t)$ in a lake is attacked by disease at time $t = 0$, with the result that

$$\frac{dP}{dt} = -k\sqrt{P} \qquad (k > 0)$$

thereafter. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long would it take all the fish in the lake to die?

22. Prove that the solution of the initial value problem

$$\frac{dP}{dt} = k\sqrt{P}, \qquad P(0) = P_0 \qquad (P_0 > 0)$$

is given by

$$P(t) = \left(\tfrac{1}{2}kt + \sqrt{P_0}\right)^2.$$

23. Suppose that the population of Fremont satisfies the differential equation of Problem 22. (a) If $P = 100{,}000$ in 1970 and $P = 121{,}000$ in 1980, what will the population be in 2000? (b) When will the population reach 200,000?

24. Consider a breed of rabbits whose population $P(t)$ satisfies the initial value problem

$$\frac{dP}{dt} = kP^2, \qquad P(0) = P_0,$$

where k is a positive constant. Derive the solution

$$P(t) = \frac{P_0}{1 - kP_0 t}.$$

25. In Problem 24, suppose that $P_0 = 2$ and that there are 4 rabbits after 3 months. What happens in the next 3 months?

26. Suppose that a motorboat is traveling at $v = 40$ ft/s when its motor is cut off at time $t = 0$. Thereafter its deceleration due to water resistance is given by $dv/dt = -kv^2$, where k is a positive constant. (a) Solve this differential equation to show that the speed of the boat at time $t > 0$ is $v = 40/(1 + 40kt)$ feet per second. (b) If the boat's speed after 10 s is 20 ft/s, how long does it take to slow to 5 ft/s?

27. A tank shaped like a vertical cylinder initially contains water to a depth of 9 ft (Fig. 6.5.6). A bottom plug is pulled at time $t = 0$ (t in hours). After 1 h the depth has dropped to 4 ft. How long will it take all the water to drain from this tank?

28. Suppose that the tank of Problem 27 has a radius of 3 ft and that its bottom hole is circular with radius 1 in. How long will it take for the water, initially 9 ft deep, to drain completely?

29. A water tank is in the shape of a right circular cone with its axis vertical and its vertex at the bottom. The tank is 16 ft high and the radius of its top is 5 ft. At time $t = 0$, a plug at its vertex is removed and the tank, initially full of water, begins to drain. After 1 h the water in the tank is 9 ft deep. When will the tank be empty (Fig. 6.5.7)?

30. Suppose that a cylindrical tank (axis vertical) initially containing V_0 liters of water drains through a bottom hole in T minutes. Use Torricelli's law to show that the volume of water in the tank after $t \leqq T$ minutes is $V(t) = V_0[1 - (t/T)]^2$.

31. The shape of a water tank is obtained by revolving the curve $y = x^{4/3}$ around the y-axis (units on the coordinate axes are in feet). A plug at the bottom is removed at 12 noon, when the water depth in the tank is 12 ft. At 1 P.M. the water depth is 6 ft. When will the tank be empty?

32. The shape of a water tank is obtained by revolving the parabola $y = x^2$ around the y-axis (units on the coordinate axes are in feet; see Fig. 6.5.8). The water depth is 4 ft at 12 noon; at that time, the plug in the circular hole at the bottom of the tank is removed. At 1 P.M. the water depth is 1 ft. (a) Find the water depth $y(t)$ after t hours. (b) When will the tank be empty? (c) What is the radius of the circular hole at the bottom?

33. A cylindrical tank of length 5 ft and radius 3 ft is situated with its axis horizontal. If a circular bottom hole of radius 1 in. is opened and the tank is initially half full of xylene, how long will it take the liquid to drain completely?

34. A spherical tank of radius 4 ft is full of mercury when a circular bottom hole of radius 1 in. is opened. How long will it be before all of the mercury drains from the tank?

35. *The Clepsydra, or Water Clock* A 12-h water clock is to be designed with the dimensions shown in Fig. 6.5.9, shaped like the surface obtained by revolving the curve $y = f(x)$ around the y-axis. What equation should this curve have, *and* what radius should the bottom hole have, so that the water level will fall at the *constant* rate of 4 in./h?

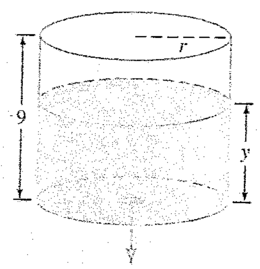

Fig. 6.5.6 The cylindrical tank of Problem 27

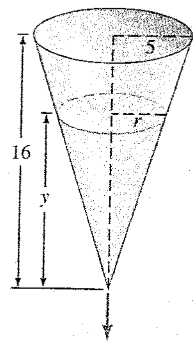

Fig. 6.5.7 The conical tank of Problem 29

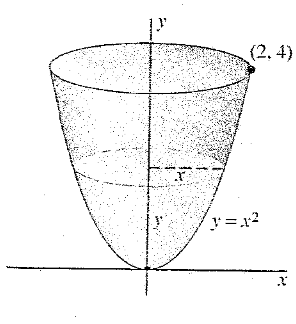

Fig. 6.5.8 The tank of Problem 32

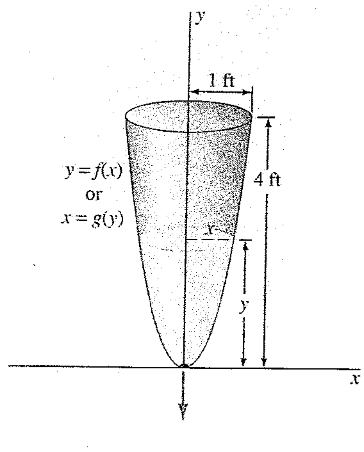

Fig. 6.5.9 The clepsydra of Problem 35

43. $y = \dfrac{3x - 15 - 2x^2}{x^3 - 9x}$, $1 \leqq x \leqq 2$

44. $y = \dfrac{x^2 + 10x + 16}{x^3 + 8x^2 + 16x}$, $2 \leqq x \leqq 5$

In Problems 45 through 48, find the volume of the solid obtained by revolving the region R around the y-axis.

45. The region R of Problem 41

46. The region R of Problem 42

47. The region R of Problem 43

48. The region R of Problem 44

In Problems 49 and 50, find the volume of the solid obtained by revolving the region R around the x-axis.

49. The region R of Problem 41

50. The region R of Problem 42

51. The plane region R shown in Fig. 9.5.6 is bounded by the curve

$$y^2 = \frac{1 - x}{1 + x}x^2, \quad 0 \leqq x \leqq 1.$$

Find the volume generated by revolving R around the x-axis.

52. Figure 9.5.7 shows the region bounded by the curve

$$y^2 = \frac{(1 - x)^2}{(1 + x)^2}x^4, \quad 0 \leqq x \leqq 1.$$

Find the volume generated by revolving this region around: (a) the x-axis; (b) the y-axis.

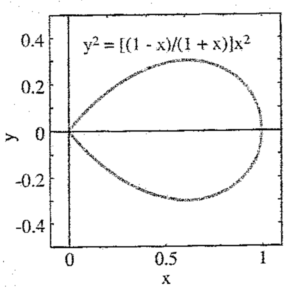

Fig. 9.5.6 The region of Problem 51

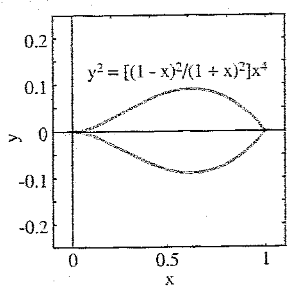

Fig. 9.5.7 The region of Problem 52

In Problems 53 through 56, write the general form of a partial fraction decomposition of the given rational function $f(x)$ (with coefficients $A, B, C, \ldots$, remaining to be determined). Then use a computer algebra system (as in the remark following Example 6) to find the numerical values of the coefficients in the decomposition. Finally, find the indefinite integral $\int f(x)\,dx$ both by hand and using the computer algebra system, and resolve any apparent discrepancy between the two results.

53. $f(x) = \dfrac{98(x^3 - 50x + 100)}{x^2(x^2 - 12x + 35)}$

54. $f(x) = \dfrac{16(2x^3 + 77x - 99)}{(x^2 + 10x + 21)^2}$

55. $f(x) = \dfrac{324(x^3 + 8)}{(x^2 - x - 6)(x^2 + x - 20)^2}$

56. $f(x) = \dfrac{500(4x^4 - 23x^2 + 16)}{(x^2 - 4)^2(x - 3)^2}$

Solve the initial value problems in 57 through 62.

57. $\dfrac{dx}{dt} = x - x^2$, $x(0) = 2$

58. $\dfrac{dx}{dt} = 10x - x^2$, $x(0) = 1$

59. $\dfrac{dx}{dt} = 1 - x^2$, $x(0) = 3$

60. $\dfrac{dx}{dt} = 9 - 4x^2$, $x(0) = 0$

61. $\dfrac{dx}{dt} = x^2 + 5x + 6$, $x(0) = 5$

62. $\dfrac{dx}{dt} = 2x^2 + x - 15$, $x(0) = 10$

63. Suppose that the population $P(t)$ (in millions) of Ruritania satisfies the differential equation

$$\frac{dP}{dt} = k \cdot P \cdot (200 - P) \qquad (k \text{ constant}).$$

Its population in 1940 was 100 million and was then growing at the rate of 1 million per year. Predict this country's population for the year 2000.

64. Suppose that a community contains 15000 people who are susceptible to Michaud's syndrome, a contagious disease. At time $t = 0$ the number $N(t)$ of people who have caught Michaud's syndrome is 5000 and is increasing at the rate of 500 per day. Assume that $N'(t)$ is proportional to the product of the numbers of those who have caught the disease and those who have not. How long will it take for another 5000 people to contract Michaud's syndrome?

65. As the salt KNO_3 dissolves in methanol, the number $x(t)$ of grams of the salt in solution after t seconds satisfies the differential equation

$$\frac{dx}{dt} = (0.8)x - (0.004)x^2.$$

(a) If $x = 50$ when $t = 0$, how long will it take an additional 50 g of the salt to dissolve? (b) What is the maximum amount of the salt that will ever dissolve in the methanol?

66. A population $P(t)$ (t in months) of squirrels satisfies the differential equation

$$\frac{dP}{dt} = (0.001)P^2 - kP \qquad (k \text{ constant}).$$