EXAMPLE 8 Figure 3.1.18 shows the graphs $y = f(x)$ of a function and $y = f'(x)$ of its derivative. Observe that

- $y = f(x)$ has a horizontal tangent line at points where $f'(x) = 0$;
- $f(x)$ is increasing on open intervals where $f'(x) > 0$; and
- $f(x)$ is decreasing on open intervals where $f'(x) < 0$.

3.1 PROBLEMS

In Problems 1 through 10, find the indicated derivative by using the differentiation rule in Eqs. (6) and (7):

$$\text{If} \quad f(x) = ax^2 + bx + c, \quad \text{then} \quad f'(x) = 2ax + b.$$

1. $f(x) = 4x - 5$; find $f'(x)$.
2. $g(t) = 100 - 16t^2$; find $g'(t)$.
3. $h(z) = z(25 - z)$; find $h'(z)$.
4. $f(x) = 16 - 49x$; find $f'(x)$.
5. $y = 2x^2 + 3x - 17$; find dy/dx.
6. $x = 16t - 100t^2$; find dx/dt.
7. $z = 5u^2 - 3u$; find dz/du.
8. $v = 5y(100 - y)$; find dv/dy.
9. $x = -5y^2 + 17y + 300$; find dx/dy.
10. $u = 7t^2 + 13t$; find du/dt.

In Problems 11 through 20, apply the definition of the derivative (as in Example 1) to find $f'(x)$.

11. $f(x) = 2x - 1$
12. $f(x) = 2 - 3x$
13. $f(x) = x^2 + 5$
14. $f(x) = 3 - 2x^2$
15. $f(x) = \dfrac{1}{2x + 1}$
16. $f(x) = \dfrac{1}{3 - x}$
17. $f(x) = \sqrt{2x + 1}$
18. $f(x) = \dfrac{1}{\sqrt{x + 1}}$
19. $f(x) = \dfrac{x}{1 - 2x}$
20. $f(x) = \dfrac{x + 1}{x - 1}$

Problems 21 through 25 give the position function $x = f(t)$ of a particle moving in a horizontal straight line. Find its location x when its velocity v is zero.

21. $x = 100 - 16t^2$
22. $x = -16t^2 + 160t + 25$
23. $x = -16t^2 + 80t - 1$
24. $x = 100t^2 + 50$
25. $x = 100 - 20t - 5t^2$

Problems 26 through 29 give the height $y(t)$ (in feet at time t seconds) of a ball thrown vertically upward. Find the maximum height that the ball attains.

26. $y = -16t^2 + 160t$
27. $y = -16t^2 + 64t$
28. $y = -16t^2 + 128t + 25$
29. $y = -16t^2 + 96t + 50$

In Problems 30 through 35 (Figs. 3.1.19 through 3.1.24), match the given graph of the function f with that of its derivative, which appears among those in Fig. 3.1.25, parts (a) through (f).

30. Figure 3.1.19
31. Figure 3.1.20
32. Figure 3.1.21
33. Figure 3.1.22
34. Figure 3.1.23
35. Figure 3.1.24

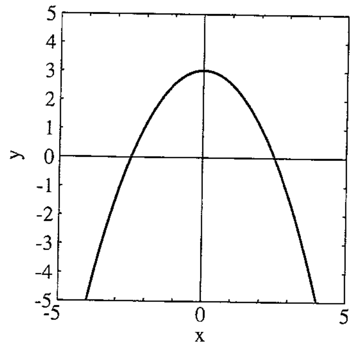

Fig. 3.1.19

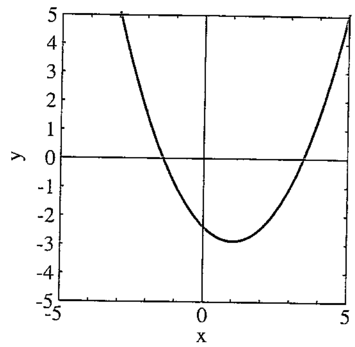

Fig. 3.1.20

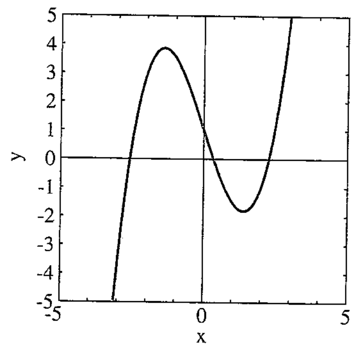

Fig. 3.1.21

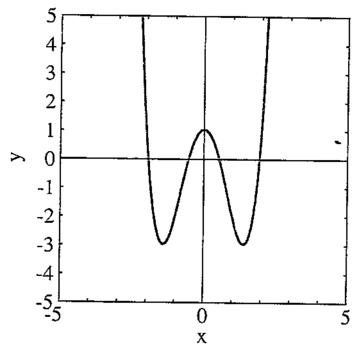

Fig. 3.1.22

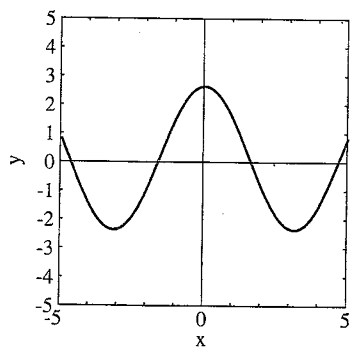

Fig. 3.1.23

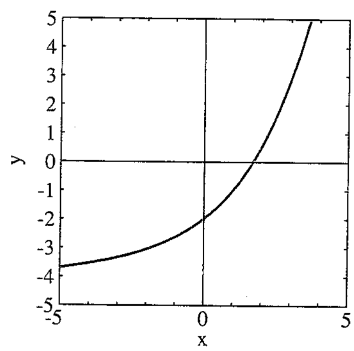

Fig. 3.1.24

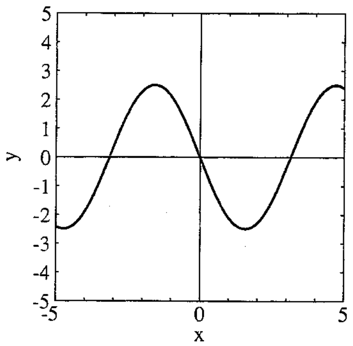

Fig. 3.1.25(a)

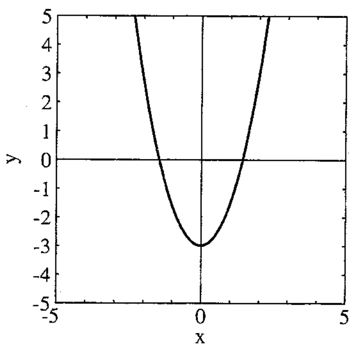

Fig. 3.1.25(b)

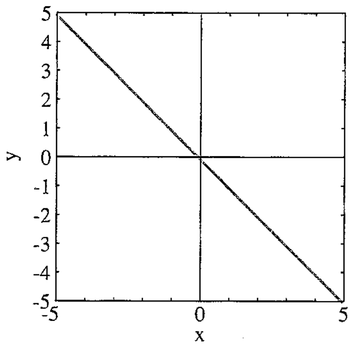

Fig. 3.1.25(c)

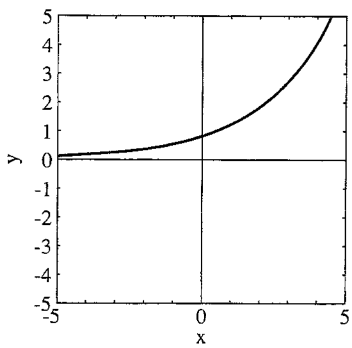

Fig. 3.1.25(d)

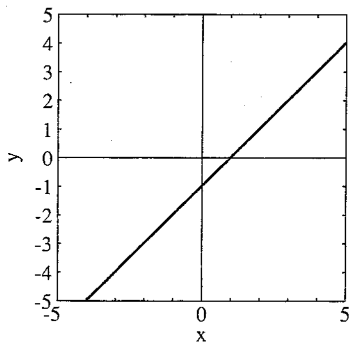

Fig. 3.1.25(e)

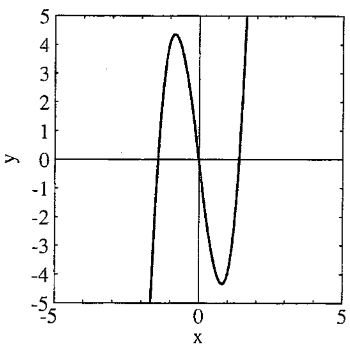

Fig. 3.1.25(f)

36. The Celsius temperature C is given in terms of the Fahrenheit temperature F by $C = \frac{5}{9}(F - 32)$. Find the rate of change of C with respect to F and the rate of change of F with respect to C.

37. Find the rate of change of the area A of a circle with respect to its circumference C.

38. A stone dropped into a pond at time $t = 0$ s causes a circular ripple that travels out from the point of impact at 5 m/s. At what rate (in square meters per second) is the area within the circle increasing when $t = 10$?

39. A car is traveling at 100 ft/s when the driver suddenly applies the brakes ($x = 0, t = 0$). The position function of the skidding car is $x(t) = 100t - 5t^2$. How far and for how long does the car skid before it comes to a stop?

40. A water bucket containing 10 gal of water develops a leak at time $t = 0$, and the volume V of water in the bucket t seconds later is given by

$$V(t) = 10\left(1 - \frac{t}{100}\right)^2$$

until the bucket is empty at time $t = 100$. (a) At what rate is water leaking from the bucket after exactly 1 min has passed? (b) When is the instantaneous rate of change of V equal to the average rate of change of V from $t = 0$ to $t = 100$?

41. A population of chipmunks moves into a new region at time $t = 0$. At time t (in months) the population numbers

$$P(t) = 100[1 + (0.3)t + (0.04)t^2].$$

(a) How long does it take for this population to double its initial size $P(0)$? (b) What is the rate of growth of the population when $P = 200$?

42. The following data describe the growth of the population P (in thousands) of Gotham City during a 10-year period. Use the graphical method of Example 4 to estimate its rate of growth in 1989.

YEAR	1984	1986	1988	1990	1992	1994
P	265	293	324	358	395	427

43. The following data give the distance x in feet traveled by an accelerating car (that starts from rest at time $t = 0$) in the first t seconds. Use the graphical method of Example 4 to estimate its speed (in miles per hour) when $t = 20$ and again when $t = 40$.

t	0	10	20	30	40	50	60
x	0	224	810	1655	2686	3850	5109

In Problems 44 through 49, use the fact (proved in Section 3.2) that the derivative of $y = ax^3 + bx^2 + cx + d$ is $dy/dx = 3ax^2 + 2bx + c$.

44. Prove that the rate of change of the volume V of a cube with respect to its edge length x is equal to half the surface area S of the cube (Fig. 3.1.26).

45. Show that the rate of change of the volume V of a sphere with respect to its radius r is equal to its surface area S (Fig. 3.1.27).

3.3 PROBLEMS

Find dy/dx in Problems 1 through 12.

1. $y = (3x + 4)^5$
2. $y = (2 - 5x)^3$
3. $y = \dfrac{1}{3x - 2}$
4. $y = \dfrac{1}{(2x + 1)^3}$
5. $y = (x^2 + 3x + 4)^3$
6. $y = (7 - 2x^3)^{-4}$
7. $y = (2 - x)^4(3 + x)^7$
8. $y = (x + x^2)^5(1 + x^3)^2$
9. $y = \dfrac{x + 2}{(3x - 4)^3}$
10. $y = \dfrac{(1 - x^2)^3}{(4 + 5x + 6x^2)^2}$
11. $y = [1 + (1 + x)^3]^4$
12. $y = [x + (x + x^2)^{-3}]^{-5}$

In Problems 13 through 20, express the derivative dy/dx in terms of x.

13. $y = (u + 1)^3$ and $u = \dfrac{1}{x^2}$
14. $y = \dfrac{1}{2u} - \dfrac{1}{3u^2}$ and $u = 2x + 1$
15. $y = (1 + u^2)^3$ and $u = (4x - 1)^2$
16. $y = u^5$ and $u = \dfrac{1}{3x - 2}$
17. $y = u(1 - u)^3$ and $u = \dfrac{1}{x^4}$
18. $y = \dfrac{u}{u + 1}$ and $u = \dfrac{x}{x + 1}$
19. $y = u^2(u - u^4)^3$ and $u = \dfrac{1}{x^2}$
20. $y = \dfrac{u}{(2u + 1)^4}$ and $u = x - \dfrac{2}{x}$

In Problems 21 through 26, identify a function u of x and an integer $n \neq 1$ such that $f(x) = u^n$. Then compute $f'(x)$.

21. $f(x) = (2x - x^2)^3$
22. $f(x) = \dfrac{1}{2 + 5x^3}$
23. $f(x) = \dfrac{1}{(1 - x^2)^4}$
24. $f(x) = (x^2 - 4x + 1)^3$
25. $f(x) = \left(\dfrac{x + 1}{x - 1}\right)^7$
26. $f(x) = \dfrac{(x^2 + x + 1)^4}{(x + 1)^4}$

Differentiate the functions given in Problems 27 through 36.

27. $g(y) = y + (2y - 3)^5$
28. $h(z) = z^2(z^2 + 4)^3$
29. $F(s) = \left(s - \dfrac{1}{s^2}\right)^3$
30. $G(t) = \left(t^2 + 1 + \dfrac{1}{t}\right)^2$
31. $f(u) = (1 + u)^3(1 + u^2)^4$
32. $g(w) = (w^2 - 3w + 4)(w + 4)^5$
33. $h(v) = \left[v - \left(1 - \dfrac{1}{v}\right)^{-1}\right]^{-2}$
34. $p(t) = \left(\dfrac{1}{t} + \dfrac{1}{t^2} + \dfrac{1}{t^3}\right)^{-4}$
35. $F(z) = \dfrac{1}{(3 - 4z + 5z^5)^{10}}$
36. $G(x) = \{1 + [x + (x^2 + x^3)^4]^5\}^6$

In Problems 37 through 44, dy/dx can be found in two ways—either using the chain rule or not using it. Use both techniques to find dy/dx, then compare the answers. (They should agree!)

37. $y = (x^3)^4 = x^{12}$
38. $y = x = \left(\dfrac{1}{x}\right)^{-1}$
39. $y = (x^2 - 1)^2 = x^4 - 2x^2 + 1$
40. $y = (1 - x)^3 = 1 - 3x + 3x^2 - x^3$
41. $y = (x + 1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$
42. $y = (x + 1)^{-2} = \dfrac{1}{x^2 + 2x + 1}$
43. $y = (x^2 + 1)^{-1} = \dfrac{1}{x^2 + 1}$
44. $y = (x^2 + 1)^2 = (x^2 + 1)(x^2 + 1)$

We shall see in Section 3.7 that $D_x[\sin x] = \cos x$ (provided that x is in radian measure). Use this fact and the chain rule to find the derivatives of the functions in Problems 45 through 48.

45. $f(x) = \sin(x^3)$
46. $g(t) = (\sin t)^3$
47. $g(z) = (\sin 2z)^3$
48. $k(u) = \sin(1 + \sin u)$
49. A pebble dropped into a lake creates an expanding circular ripple (Fig. 3.3.4). Suppose that the radius of the circle is increasing at the rate of 2 in./s. At what rate is its area increasing when its radius is 10 in.?
50. The area of a circle is decreasing at the rate of 2π cm^2/s. At what rate is the radius of the circle decreasing when its area is 75π cm^2?
51. Each edge x of a square is increasing at the rate of 2 in./s. At what rate is the area A of the square increasing when each edge is 10 in.?
52. Each edge of an equilateral triangle is increasing at 2 cm/s (Fig. 3.3.5). At what rate is the area of the triangle increasing when each edge is 10 cm?
53. A cubical block of ice is melting in such a way that each edge decreases steadily by 2 in. every hour. At what rate is its volume decreasing when each edge is 10 in. long?

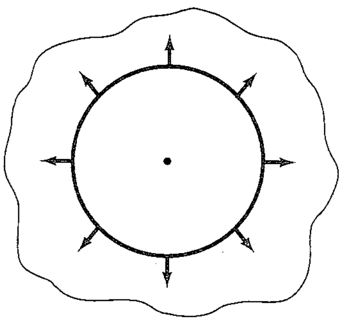

Fig. 3.3.4 Expanding circular ripple in a lake (Problem 49)

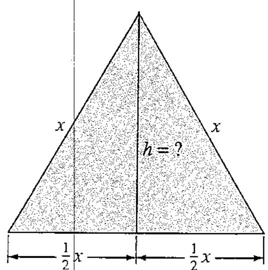

Fig. 3.3.5 The triangle of Problem 52 with area $A = \frac{1}{2}xh$

54. Find $f'(-1)$, given $f(y) = h(g(y)), h(2) = 55, g(-1) = 2,$ $h'(2) = -1,$ and $g'(-1) = 7$.

55. Given: $G(t) = f(h(t)),\quad h(1) = 4,\quad f'(4) = 3,$ and $h'(1) = -6$. Find $G'(1)$.

56. Suppose that $f(0) = 0$ and that $f'(0) = 1$. Calculate the derivative of $f(f(f(x)))$ at $x = 0$.

57. Air is being pumped into a spherical balloon in such a way that its radius r is increasing at the rate of $dr/dt = 1$ cm/s. What is the time rate of increase, in cubic centimeters per second, of the balloon's volume when $r = 10$ cm?

58. Suppose that the air is being pumped into the balloon of Problem 57 at the constant rate of 200π cm^3/s. What is the time rate of increase of the radius r when $r = 5$ cm?

59. Air is escaping from a spherical balloon at the constant rate of 300π cm^3/s. What is the radius of the balloon when its radius is decreasing at the rate of 3 cm/s?

60. A spherical hailstone is losing mass by melting uniformly over its surface as it falls. At a certain time, its radius is 2 cm and its volume is decreasing at the rate of 0.1 cm^3/s. How fast is its radius decreasing at that time?

61. A spherical snowball is melting in such a way that the rate of decrease of its volume is proportional to its surface area. At 10 A.M. its volume is 500 in.^{3} and at 11 A.M. its volume is 250 in.^{3}. When does the snowball finish melting? (See Example 7.)

62. A cubical block of ice with edges 20 in. long begins to melt at 8 A.M. Each edge decreases at a constant rate thereafter and each is 8 in. long at 4 P.M. What was the rate of change of the block's volume at noon?

63. Suppose that u is a function of v, that v is a function of w, that w is a function of x, and that all these functions are differentiable. Explain why it follows from the chain rule that

$$\frac{du}{dx} = \frac{du}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{dx}.$$

64. Let f be a differentiable function such that $f(1) = 1$. If $F(x) = f(x^n)$ and $G(x) = [f(x)]^n$ (where n is a fixed integer), show that $F(1) = G(1)$ and that $F'(1) = G'(1)$.

3.4 DERIVATIVES OF ALGEBRAIC FUNCTIONS

We saw in Section 3.3 that the chain rule yields the differentiation formula

$$D_x\, u^n = nu^{n-1}\frac{du}{dx} \tag{1}$$

if $u = f(x)$ is a differentiable function and the exponent n is an integer. We shall see in Theorem 1 of this section that this **generalized power rule** holds not only when the exponent is an integer, but also when it is a rational number $r = p/q$ (where p and q are integers and $q \neq 0$). Recall that rational powers are defined in terms of integral roots and powers as follows:

$$u^{p/q} = \sqrt[q]{u^p} = \left(\sqrt[q]{u}\right)^p.$$

We first consider the case of a rational power of the independent variable x:

$$y = x^{p/q}, \tag{2}$$

where p and q are integers with q positive. We show independently in Section 7.4 that $g(x) = x^{p/q}$ is differentiable wherever its derivative does not involve division by zero or an even root of a negative number. Assuming this fact, let us take the qth power of each side in Eq. (2) to obtain

$$y^q = x^p \tag{3}$$

[because $(x^{p/q})^q = x^p$]. Note that Eq. (3) is an identity—the functions y^q and x^p of x are identical where defined. Therefore their derivatives with respect to x must also be identical. That is,

$$D_x(y^q) = D_x(x^p);$$

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}.$$