3.3 PROBLEMS

Find dy/dx in Problems 1 through 12.

1. $y = (3x + 4)^5$
2. $y = (2 - 5x)^3$
3. $y = \dfrac{1}{3x - 2}$
4. $y = \dfrac{1}{(2x + 1)^3}$
5. $y = (x^2 + 3x + 4)^3$
6. $y = (7 - 2x^3)^{-4}$
7. $y = (2 - x)^4(3 + x)^7$
8. $y = (x + x^2)^5(1 + x^3)^2$
9. $y = \dfrac{x + 2}{(3x - 4)^3}$
10. $y = \dfrac{(1 - x^2)^3}{(4 + 5x + 6x^2)^2}$
11. $y = [1 + (1 + x)^3]^4$
12. $y = [x + (x + x^2)^{-3}]^{-5}$

In Problems 13 through 20, express the derivative dy/dx in terms of x.

13. $y = (u + 1)^3$ and $u = \dfrac{1}{x^2}$
14. $y = \dfrac{1}{2u} - \dfrac{1}{3u^2}$ and $u = 2x + 1$
15. $y = (1 + u^2)^3$ and $u = (4x - 1)^2$
16. $y = u^5$ and $u = \dfrac{1}{3x - 2}$
17. $y = u(1 - u)^3$ and $u = \dfrac{1}{x^4}$
18. $y = \dfrac{u}{u + 1}$ and $u = \dfrac{x}{x + 1}$
19. $y = u^2(u - u^4)^3$ and $u = \dfrac{1}{x^2}$
20. $y = \dfrac{u}{(2u + 1)^4}$ and $u = x - \dfrac{2}{x}$

In Problems 21 through 26, identify a function u of x and an integer $n \neq 1$ such that $f(x) = u^n$. Then compute $f'(x)$.

21. $f(x) = (2x - x^2)^3$
22. $f(x) = \dfrac{1}{2 + 5x^3}$
23. $f(x) = \dfrac{1}{(1 - x^2)^4}$
24. $f(x) = (x^2 - 4x + 1)^3$
25. $f(x) = \left(\dfrac{x + 1}{x - 1}\right)^7$
26. $f(x) = \dfrac{(x^2 + x + 1)^4}{(x + 1)^4}$

Differentiate the functions given in Problems 27 through 36.

27. $g(y) = y + (2y - 3)^5$
28. $h(z) = z^2(z^2 + 4)^3$
29. $F(s) = \left(s - \dfrac{1}{s^2}\right)^3$
30. $G(t) = \left(t^2 + 1 + \dfrac{1}{t}\right)^2$
31. $f(u) = (1 + u)^3(1 + u^2)^4$
32. $g(w) = (w^2 - 3w + 4)(w + 4)^5$
33. $h(v) = \left[v - \left(1 - \dfrac{1}{v}\right)^{-1}\right]^{-2}$
34. $p(t) = \left(\dfrac{1}{t} + \dfrac{1}{t^2} + \dfrac{1}{t^3}\right)^{-4}$
35. $F(z) = \dfrac{1}{(3 - 4z + 5z^5)^{10}}$
36. $G(x) = \{1 + [x + (x^2 + x^3)^4]^5\}^6$

In Problems 37 through 44, dy/dx can be found in two ways—either using the chain rule or not using it. Use both techniques to find dy/dx, then compare the answers. (They should agree!)

37. $y = (x^3)^4 = x^{12}$
38. $y = x = \left(\dfrac{1}{x}\right)^{-1}$
39. $y = (x^2 - 1)^2 = x^4 - 2x^2 + 1$
40. $y = (1 - x)^3 = 1 - 3x + 3x^2 - x^3$
41. $y = (x + 1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$
42. $y = (x + 1)^{-2} = \dfrac{1}{x^2 + 2x + 1}$
43. $y = (x^2 + 1)^{-1} = \dfrac{1}{x^2 + 1}$
44. $y = (x^2 + 1)^2 = (x^2 + 1)(x^2 + 1)$

We shall see in Section 3.7 that $D_x[\sin x] = \cos x$ (provided that x is in radian measure). Use this fact and the chain rule to find the derivatives of the functions in Problems 45 through 48.

45. $f(x) = \sin(x^3)$
46. $g(t) = (\sin t)^3$
47. $g(z) = (\sin 2z)^3$
48. $k(u) = \sin(1 + \sin u)$

49. A pebble dropped into a lake creates an expanding circular ripple (Fig. 3.3.4). Suppose that the radius of the circle is increasing at the rate of 2 in./s. At what rate is its area increasing when its radius is 10 in.?

50. The area of a circle is decreasing at the rate of 2π cm^2/s. At what rate is the radius of the circle decreasing when its area is 75π cm^2?

51. Each edge x of a square is increasing at the rate of 2 in./s. At what rate is the area A of the square increasing when each edge is 10 in.?

52. Each edge of an equilateral triangle is increasing at 2 cm/s (Fig. 3.3.5). At what rate is the area of the triangle increasing when each edge is 10 cm?

53. A cubical block of ice is melting in such a way that each edge decreases steadily by 2 in. every hour. At what rate is its volume decreasing when each edge is 10 in. long?

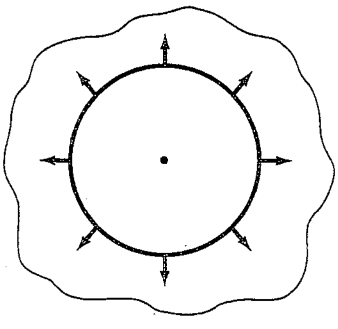

Fig. 3.3.4 Expanding circular ripple in a lake (Problem 49)

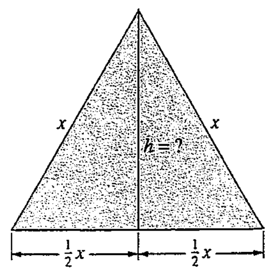

Fig. 3.3.5 The triangle of Problem 52 with area $A = \frac{1}{2}xh$

54. Find $f'(-1)$, given $f(y) = h(g(y))$, $h(2) = 55$, $g(-1) = 2$, $h'(2) = -1$, and $g'(-1) = 7$.

55. Given: $G(t) = f(h(t))$, $h(1) = 4$, $f'(4) = 3$, and $h'(1) = -6$. Find $G'(1)$.

56. Suppose that $f(0) = 0$ and that $f'(0) = 1$. Calculate the derivative of $f(f(f(x)))$ at $x = 0$.

57. Air is being pumped into a spherical balloon in such a way that its radius r is increasing at the rate of $dr/dt = 1$ cm/s. What is the time rate of increase, in cubic centimeters per second, of the balloon's volume when $r = 10$ cm?

58. Suppose that the air is being pumped into the balloon of Problem 57 at the constant rate of 200π cm^3/s. What is the time rate of increase of the radius r when $r = 5$ cm?

59. Air is escaping from a spherical balloon at the constant rate of 300π cm^3/s. What is the radius of the balloon when its radius is decreasing at the rate of 3 cm/s?

60. A spherical hailstone is losing mass by melting uniformly over its surface as it falls. At a certain time, its radius is 2 cm and its volume is decreasing at the rate of 0.1 cm^3/s. How fast is its radius decreasing at that time?

61. A spherical snowball is melting in such a way that the rate of decrease of its volume is proportional to its surface area. At 10 A.M. its volume is 500 in.^{3} and at 11 A.M. its volume is 250 in.^{3}. When does the snowball finish melting? (See Example 7.)

62. A cubical block of ice with edges 20 in. long begins to melt at 8 A.M. Each edge decreases at a constant rate thereafter and each is 8 in. long at 4 P.M. What was the rate of change of the block's volume at noon?

63. Suppose that u is a function of v, that v is a function of w, that w is a function of x, and that all these functions are differentiable. Explain why it follows from the chain rule that

$$\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}.$$

64. Let f be a differentiable function such that $f(1) = 1$. If $F(x) = f(x^n)$ and $G(x) = [f(x)]^n$ (where n is a fixed integer), show that $F(1) = G(1)$ and that $F'(1) = G'(1)$.

3.4 DERIVATIVES OF ALGEBRAIC FUNCTIONS

We saw in Section 3.3 that the chain rule yields the differentiation formula

$$D_x u^n = nu^{n-1} \frac{du}{dx} \tag{1}$$

if $u = f(x)$ is a differentiable function and the exponent n is an integer. We shall see in Theorem 1 of this section that this **generalized power rule** holds not only when the exponent is an integer, but also when it is a rational number $r = p/q$ (where p and q are integers and $q \neq 0$). Recall that rational powers are defined in terms of integral roots and powers as follows:

$$u^{p/q} = \sqrt[q]{u^p} = \left(\sqrt[q]{u}\right)^p.$$

We first consider the case of a rational power of the independent variable x:

$$y = x^{p/q}, \tag{2}$$

where p and q are integers with q positive. We show independently in Section 7.4 that $g(x) = x^{p/q}$ is differentiable wherever its derivative does not involve division by zero or an even root of a negative number. Assuming this fact, let us take the qth power of each side in Eq. (2) to obtain

$$y^q = x^p \tag{3}$$

[because $(x^{p/q})^q = x^p$]. Note that Eq. (3) is an identity—the functions y^q and x^p of x are identical where defined. Therefore their derivatives with respect to x must also be identical. That is,

$$D_x(y^q) = D_x(x^p);$$

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}.$$

+62 P. 139.   Ex. 57 – 62 ~~set~~ match with 3.4, 3.4.13 a), b)

38. $h(z) = (z - 1)^4(z + 1)^6$

39. $f(x) = \dfrac{(2x + 1)^{1/2}}{(3x + 4)^{1/3}}$

40. $f(x) = (1 - 3x^4)^5(4 - x)^{1/3}$

41. $h(y) = \dfrac{\sqrt{1 + y} + \sqrt{1 - y}}{\sqrt[3]{y^5}}$

42. $f(x) = \sqrt{1 - \sqrt[3]{x}}$

43. $g(t) = \sqrt{t + \sqrt{t + \sqrt{t}}}$ **44.** $f(x) = x^3\sqrt{1 - \dfrac{1}{x^2 + 1}}$

For each curve given in Problems 45 through 50, find all points on the graph where the tangent line is either horizontal or vertical.

45. $y = x^{2/3}$ **46.** $y = x\sqrt{4 - x^2}$

47. $y = x^{1/2} - x^{3/2}$ **48.** $y = \dfrac{1}{\sqrt{9 - x^2}}$

49. $y = \dfrac{x}{\sqrt{1 - x^2}}$

50. $y = \sqrt{(1 - x^2)(4 - x^2)}$

In Problems 51 through 56, first write an equation of the line tangent to the given curve $y = f(x)$ at the indicated point P. Then illustrate your result with a graphing calculator or computer by graphing both the curve and the tangent line on the same screen.

51. $y = 2\sqrt{x}$, at the point P where $x = 4$

52. $y = 3\sqrt[3]{x}$, at the point P where $x = 8$

53. $y = 3\sqrt[3]{x^2}$, at the point P where $x = -1$

54. $y = 2\sqrt{1 - x}$, at the point P where $x = \frac{3}{4}$

55. $y = x\sqrt{4 - x}$, at the point P where $x = 0$

56. $y = (1 - x)\sqrt{x}$, at the point P where $x = 4$

In Problems 57 through 62 (Figs. 3.4.7 through 3.4.12), match the graph $y = f(x)$ of a function with the graph $y = f'(x)$ of its derivative among those shown in Figs. 3.4.13(a) through 3.4.13(f).

57. Figure 3.4.7 **58.** Figure 3.4.8

59. Figure 3.4.9 **60.** Figure 3.4.10

61. Figure 3.4.11 **62.** Figure 3.4.12

63. The period of oscillation P (in seconds) of a simple pendulum of length L (in feet) is given by $P = 2\pi\sqrt{L/g}$, where $g = 32$ ft/s^2. Find the rate of change of P with respect to L when $P = 2$.

64. Find the rate of change of the volume $V = \frac{4}{3}\pi r^3$ of a sphere of radius r with respect to its surface area $A = 4\pi r^2$ when $r = 10$.

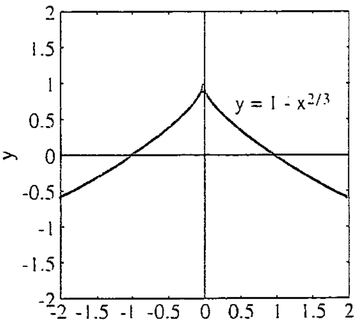

Fig. 3.4.9 $y = 1 - x^{2/3}$ (Problem 59)

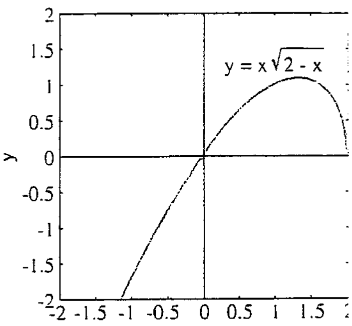

Fig. 3.4.10 $y = x\sqrt{2 - x}$ (Problem 60)

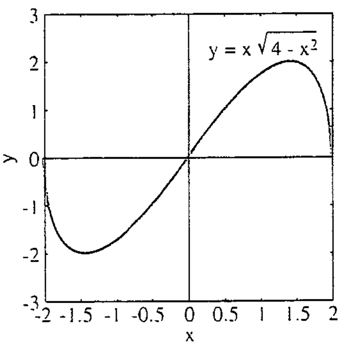

Fig. 3.4.11 $y = x\sqrt{4 - x^2}$ (Problem 61)

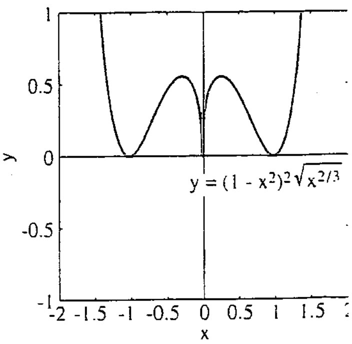

Fig. 3.4.12 $y = (1 - x^2)^2\sqrt{x^{2/3}}$ (Problem 62)

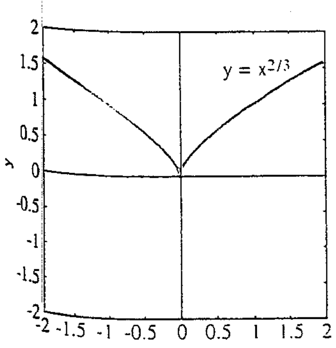

Fig. 3.4.7 $y = x^{2/3}$ (Problem 57)

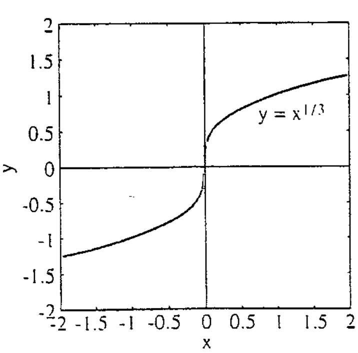

Fig. 3.4.8 $y = x^{1/3}$ (Problem 58)

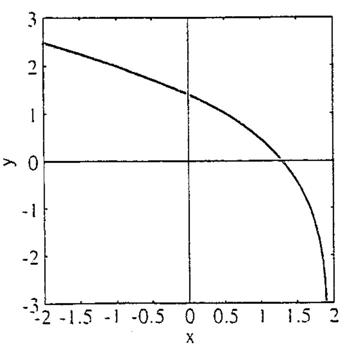

Fig. 3.4.13 (a)

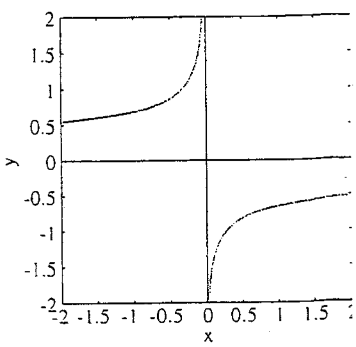

Fig. 3.4.13 (b)

Planet	T(in days)	a (in 10^6 km)	ln T	ln a
Mercury	87.97	58	4.48	4.06
Venus	224.70	108	5.41	4.68
Earth	365.26	149	5.90	5.00
Mars	686.98	228	6.53	5.43
Jupiter	4332.59	778	8.37	6.66
Saturn	10,759.20	1426	9.28	7.26

Fig. 7.2.12 Data for Example 9

$\ln T$ against $\ln a$, it is immediately apparent that the resulting points lie on a straight line of slope $m = \frac{3}{2}$. Hence T and a satisfy an equation of the form $T = ka^{3/2}$, so

$$T^2 = Ca^3.$$

This means that the square of the period T is proportional to the cube of the major semiaxis a. This is Kepler's third law of planetary motion, which Johannes Kepler (1571–1630) discovered empirically in 1619. ■

7.2 PROBLEMS

Differentiate the functions given in Problems 1 through 18.

1. $f(x) = \ln(3x - 1)$
2. $f(x) = \ln(4 - x^2)$
3. $f(x) = \ln\sqrt{1 + 2x}$
4. $f(x) = \ln[(1 + x)^2]$
5. $f(x) = \ln\sqrt[3]{x^3 - x}$
6. $f(x) = \ln(\sin^2 x)$
7. $f(x) = \cos(\ln x)$
8. $f(x) = (\ln x)^3$
9. $f(x) = \dfrac{1}{\ln x}$
10. $f(x) = \ln(\ln x)$
11. $f(x) = \ln(x\sqrt{x^2 + 1})$
12. $g(t) = t^{3/2}\ln(t + 1)$
13. $f(x) = \ln(\cos x)$
14. $f(x) = \ln(2\sin x)$
15. $f(t) = t^2\ln(\cos t)$
16. $f(x) = \sin(\ln 2x)$
17. $g(t) = t(\ln t)^2$
18. $g(t) = \sqrt{t}\,[\cos(\ln t)]^2$

In Problems 19 through 28, apply laws of logarithms to simplify the given function; then write its derivative.

19. $f(x) = \ln[(2x + 1)^3(x^2 - 4)^4]$
20. $f(x) = \ln\sqrt{\dfrac{1 - x}{1 + x}}$
21. $f(x) = \ln\sqrt{\dfrac{4 - x^2}{9 + x^2}}$
22. $f(x) = \ln\dfrac{\sqrt{4x - 7}}{(3x - 2)^3}$
23. $f(x) = \ln\dfrac{x + 1}{x - 1}$
24. $f(x) = x^2\ln\dfrac{1}{2x + 1}$
25. $g(t) = \ln\dfrac{t^2}{t^2 + 1}$
26. $f(x) = \ln\dfrac{\sqrt{x + 1}}{(x - 1)^3}$
27. $f(x) = \ln\dfrac{\sin x}{x}$
28. $f(x) = \ln\dfrac{\sin x}{\cos x}$

In Problems 29 through 32, find dy/dx by implicit differentiation.

29. $y = x\ln y$
30. $y = (\ln x)(\ln y)$
31. $xy = \ln(\sin y)$
32. $xy + x^2(\ln y)^2 = 4$

Evaluate the indefinite integrals in Problems 33 through 50.

33. $\displaystyle\int \frac{dx}{2x - 1}$
34. $\displaystyle\int \frac{dx}{3x + 5}$
35. $\displaystyle\int \frac{x}{1 + 3x^2}\,dx$
36. $\displaystyle\int \frac{x^2}{4 - x^3}\,dx$
37. $\displaystyle\int \frac{x + 1}{2x^2 + 4x + 1}\,dx$
38. $\displaystyle\int \frac{\cos x}{1 + \sin x}\,dx$
39. $\displaystyle\int \frac{1}{x}(\ln x)^2\,dx$
40. $\displaystyle\int \frac{1}{x\ln x}\,dx$
41. $\displaystyle\int \frac{1}{x + 1}\,dx$
42. $\displaystyle\int \frac{x}{1 - x^2}\,dx$
43. $\displaystyle\int \frac{2x + 1}{x^2 + x + 1}\,dx$
44. $\displaystyle\int \frac{x + 1}{x^2 + 2x + 3}\,dx$
45. $\displaystyle\int \frac{\ln x}{x}\,dx$
46. $\displaystyle\int \frac{\ln(x^3)}{x}\,dx$
47. $\displaystyle\int \frac{\sin 2x}{1 - \cos 2x}\,dx$
48. $\displaystyle\int \frac{dx}{x(\ln x)^2}$
49. $\displaystyle\int \frac{x^2 - 2x}{x^3 - 3x^2 + 1}\,dx$
50. $\displaystyle\int \frac{dx}{\sqrt{x}(1 + \sqrt{x})}$ (*Suggestion:* Let $y = 1 + \sqrt{x}$.)

Apply Eq. (13) to evaluate the limits in Problems 51 through 56.

51. $\displaystyle\lim_{x\to\infty} \frac{\ln\sqrt{x}}{x}$
52. $\displaystyle\lim_{x\to\infty} \frac{\ln(x^3)}{x^2}$
53. $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ (*Suggestion:* Substitute $x = u^2$.)