We differentiate implicitly the two equations in (8), and we obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2u\frac{du}{dt} \quad \text{and} \quad -2(6-x)\frac{dx}{dt} + 2y\frac{dy}{dt} = 2v\frac{dv}{dt}.$$

When we substitute the numerical data given and data deduced, we find that

$$3\frac{dx}{dt} + 4\frac{dy}{dt} = 140 \quad \text{and} \quad -3\frac{dx}{dt} + 4\frac{dy}{dt} = 20.$$

These equations are easy to solve: $dx/dt = dy/dt = 20$. Therefore, the ship is sailing northeast at a speed of

$$\sqrt{20^2 + 20^2} = 20\sqrt{2} \quad \text{(km/h)}$$

—*if* the figure is correct! A mirror along the line AB will reflect *another* ship, 3 km east and 4 km *south* of A, sailing *southeast* at a speed of $20\sqrt{2}$ km/h.

The lesson? Figures are important, helpful, often essential—but potentially misleading. Avoid taking anything for granted when you draw a figure. In this example there would be no real problem, for each radar station could determine whether the ship was generally to the north or to the south.

3.8 PROBLEMS

In Problems 1 through 4, first find the derivative dy/dx by implicit differentiation. Then solve the original equation for y explicitly in terms of x and differentiate to find dy/dx. Finally verify that your two results are the same by substitution of the explicit expression for $y(x)$ in the implicit form of the derivative.

1. $x^2 - y^2 = 1$

2. $xy = 1$

3. $16x^2 + 25y^2 = 400$

4. $x^3 + y^3 = 1$

In Problems 5 through 14, find dy/dx by implicit differentiation.

5. $\sqrt{x} + \sqrt{y} = 1$

6. $x^4 + x^2y^2 + y^4 = 48$

7. $x^{2/3} + y^{2/3} = 1$

8. $(x-1)y^2 = x + 1$

9. $x^2(x - y) = y^2(x + y)$

10. $x^5 + y^5 = 5x^2y^2$

11. $x \sin y + y \sin x = 1$

12. $\cos(x + y) = \sin x \sin y$

13. $\cos^3 x + \cos^3 y = \sin(x + y)$

14. $xy = \tan xy$

In Problems 15 through 28, use implicit differentiation to find an equation of the line tangent to the given curve at the given point.

15. $x^2 + y^2 = 25$; $(3, -4)$

16. $xy = -8$; $(4, -2)$

17. $x^2y = x + 2$; $(2, 1)$

18. $x^{1/4} + y^{1/4} = 4$; $(16, 16)$

19. $xy^2 + x^2y = 2$; $(1, -2)$

20. $\dfrac{1}{x+1} + \dfrac{1}{y+1} = 1$; $(1, 1)$

21. $12(x^2 + y^2) = 25xy$; $(3, 4)$

22. $x^2 + xy + y^2 = 7$; $(3, -2)$

23. $\dfrac{1}{x^3} + \dfrac{1}{y^3} = 2$; $(1, 1)$

24. $(x^2 + y^2)^3 = 8x^2y^2$; $(1, -1)$

25. $x^{2/3} + y^{2/3} = 5$; $(8, 1)$ (Fig. 3.8.9)

26. $x^2 - xy + y^2 = 19$; $(3, -2)$ (Fig. 3.8.10)

27. $(x^2 + y^2)^2 = 50xy$; $(2, 4)$ (Fig. 3.8.11)

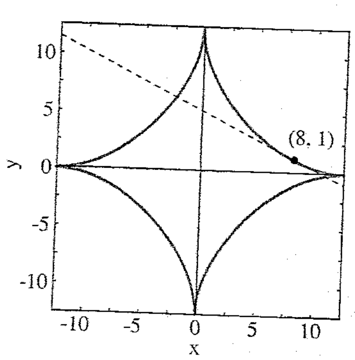

Fig. 3.8.9 Problem 25

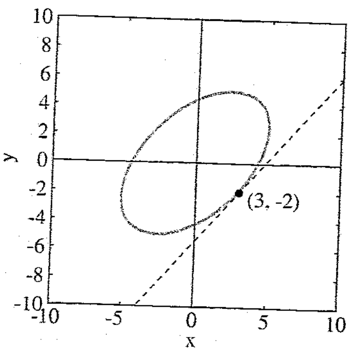

Fig. 3.8.10 Problem 26

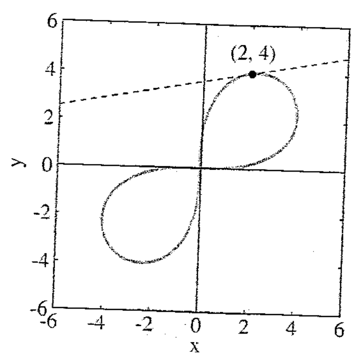

Fig. 3.8.11 Problem 27

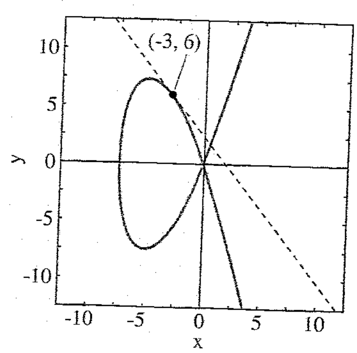

Fig. 3.8.12 Problem 28

28. $y^2 = x^2(x + 7)$; $(-3, 6)$ (Fig. 3.8.12)

29. The curve $x^3 + y^3 = 9xy$ is similar in shape and appearance to the folium of Descartes in Fig. 3.8.4. Find (a) the equation of its tangent line at the point $(2, 4)$ and (b) the equation of its tangent line with slope -1.

30. (a) Factor the left-hand side of the equation

$$2x^2 - 5xy + 2y^2 = 0$$

to show that its graph consists of two straight lines through the origin. Hence the derivative $y'(x)$ has only two possible numerical values (the slopes of these two lines). (b) Calculate dy/dx by implicit differentiation of the equation in part (a). Verify that the expression you obtain yields the proper slope for each of the straight lines of part (a).

31. Find all points on the graph of $x^2 + y^2 = 4x + 4y$ at which the tangent line is horizontal.

32. Find the first-quadrant points of the folium of Example 2 at which the tangent line is either horizontal $(dy/dx = 0)$ or vertical [where $dx/dy = 1/(dy/dx) = 0$].

33. The graph of the equation $x^2 - xy + y^2 = 9$ is the rotated ellipse shown in Fig. 3.8.13. Find the lines tangent to this curve at the two points where it intersects the x-axis, and show that these lines are parallel.

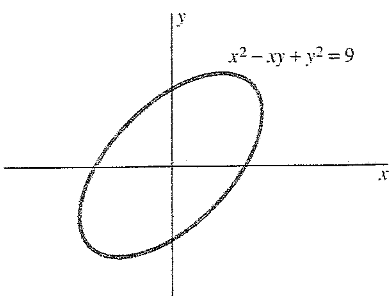

Fig. 3.8.13 The rotated ellipse of Problem 33

34. Find the points on the curve of Problem 33 where the tangent line is horizontal $(dy/dx = 0)$ and those where it is vertical $(dx/dy = 0)$.

35. The graph in Fig. 3.8.14 is a *lemniscate* with equation $(x^2 + y^2)^2 = x^2 - y^2$. Find by implicit differentiation the four points on the lemniscate where the tangent line is horizontal. Then find the two points where the tangent line is vertical—that is, where $dx/dy = 1/(dy/dx) = 0$.

36. Water is being collected from a block of ice with a square base (Fig. 3.8.15). The water is produced because the ice is melting in such a way that each edge of the base of the block is decreasing at 2 in./h while the height of the block is decreasing at 3 in./h. What is the rate of flow of water into the collecting pan when the base has edge length 20 in. and the height of the block is 15 in.? Make the simplifying assumption that water and ice have the same density.

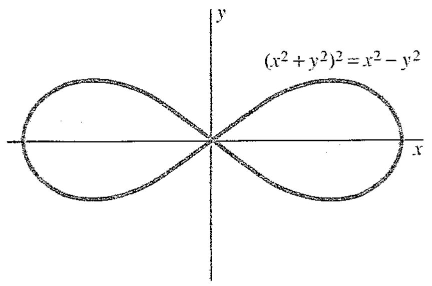

Fig. 3.8.14 The lemniscate of Problem 35

Fig. 3.8.15 The block of Problem

37. Sand is being emptied from a hopper at the rate of 10 ft^3/s. The sand forms a conical pile whose height is always twice its radius (Fig. 3.8.16). At what rate is the radius of the pile increasing when its height is 5 ft?

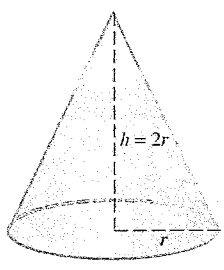

Fig. 3.8.16 The conical sand pile of Problem 37 with volume $V = \frac{1}{3}\pi r^2 h$

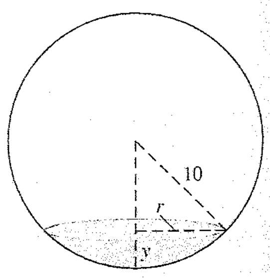

Fig. 3.8.17 The spherical tank of Problem 38

38. Suppose that water is being emptied from a spherical tank of radius 10 ft (Fig. 3.8.17). If the depth of the water in the tank is 5 ft and is decreasing at the rate of 3 ft/s, at what rate is the radius r of the top surface of the water decreasing?

39. A circular oil slick of uniform thickness is caused by a spill of 1 m^3 of oil. The thickness of the oil slick is decreasing at the rate of 0.1 cm/h. At what rate is the radius of the slick increasing when the radius is 8 m?

40. Suppose that an ostrich 5 ft tall is walking at a speed of 4 ft/s directly toward a street light 10 ft high. How fast is the tip of the ostrich's shadow moving along the ground? At what rate is the ostrich's shadow decreasing in length?

41. The width of a rectangle is half its length. At what rate is its area increasing if its width is 10 cm and is increasing at 0.5 cm/s?

42. At what rate is the area of an equilateral triangle increasing if its base is 10 cm long and is increasing at 0.5 cm/s?

43. A gas balloon is being filled at the rate of 100π cm^3 of gas per second. At what rate is the radius of the balloon increasing when its radius is 10 cm?

4.2 PROBLEMS

In Problems 1 through 16, write dy in terms of x and dx.

1. $y = 3x^2 - \dfrac{4}{x^2}$
2. $y = 2\sqrt{x} - \dfrac{3}{\sqrt[3]{x}}$
3. $y = x - \sqrt{4 - x^3}$
4. $y = \dfrac{1}{x - \sqrt{x}}$
5. $y = 3x^2(x - 3)^{3/2}$
6. $y = \dfrac{x}{x^2 - 4}$
7. $y = x(x^2 + 25)^{1/4}$
8. $y = \dfrac{1}{(x^2 - 1)^{4/3}}$
9. $y = \cos\sqrt{x}$
10. $y = x^2 \sin x$
11. $y = \sin 2x \cos 2x$
12. $y = \cos^3 3x$
13. $y = \dfrac{\sin 2x}{3x}$
14. $y = \dfrac{\cos x}{\sqrt{x}}$
15. $y = \dfrac{1}{1 - x\sin x}$
16. $y = (1 + \cos 2x)^{3/2}$

In Problems 17 through 24, find—as in Example 1—the linear approximation $L(x)$ to the given function $f(x)$ near the point $a = 0$.

17. $f(x) = \dfrac{1}{1 - x}$
18. $f(x) = \dfrac{1}{\sqrt{1 + x}}$
19. $f(x) = (1 + x)^2$
20. $f(x) = (1 - x)^3$
21. $f(x) = (1 - 2x)^{3/2}$
22. $f(x) = \dfrac{1}{(1 + 3x)^{2/3}}$
23. $f(x) = \sin x$
24. $f(x) = \cos x$

In Problems 25 through 34, use—as in Example 2—a linear approximation $L(x)$ to an appropriate function $f(x)$, with an appropriate value of a, to estimate the given number.

25. $\sqrt[3]{25}$
26. $\sqrt{102}$
27. $\sqrt[4]{15}$
28. $\sqrt{80}$
29. $65^{-2/3}$
30. $80^{3/4}$
31. $\cos 43^\circ$
32. $\sin 32^\circ$
33. $\sin 88^\circ$
34. $\cos 62^\circ$

In Problems 35 through 38, compute the differential of each side of the given equation, regarding x and y as dependent variables (as though both were functions of some third, unspecified, variable). Then solve for dy/dx.

35. $x^2 + y^2 = 1$
36. $x^{2/3} + y^{2/3} = 4$
37. $x^3 + y^3 = 3xy$
38. $x \sin y = 1$
39. Assuming that $D_x x^k = kx^{k-1}$ for any real constant k (which we shall establish in Chapter 7), derive the linear approximation formula $(1 + x)^k \approx 1 + kx$ for x near zero.

In Problems 40 through 47, use linear approximations to estimate the change in the given quantity.

40. The circumference of a circle, if its radius is increased from 10 in. to 10.5 in.
41. The area of a square, if its edge length is decreased from 10 in. to 9.8 in.
42. The surface area of a sphere, if its radius is increased from 5 in. to 5.2 in. (Fig. 4.2.13)

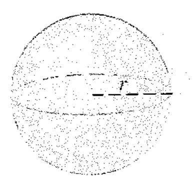

Fig. 4.2.13 The sphere of Problem 42: area $A = 4\pi r^2$, volume $V = \frac{4}{3}\pi r^3$

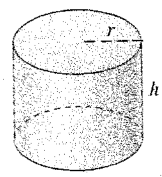

Fig. 4.2.14 The cylinder of Problem 43: volume $V = \pi r^2 h$

43. The volume of a cylinder, if both its height and its radius are decreased from 15 cm to 14.7 cm (Fig. 4.2.14)
44. The volume of the conical sandpile of Fig. 4.2.15, if its radius is 14 in. and its height is increased from 7 in. to 7.1 in.

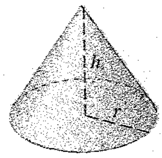

Fig. 4.2.15 The conical sandpile of Problem 44: volume $V = \frac{1}{3}\pi r^2 h$

45. The range $R = \frac{1}{16}v^2 \sin 2\theta$ of a shell fired at inclination angle $\theta = 45^\circ$, if its initial velocity v is increased from 80 ft/s to 81 ft/s
46. The range $R = \frac{1}{16}v^2 \sin 2\theta$ of a projectile fired with initial velocity $v = 80$ ft/s, if its initial inclination angle θ is increased from 45° to 46°
47. The wattage $W = RI^2$ of a floodlight with resistance $R = 10$ ohms, if the current I is increased from 3 amperes to 3.1 amperes
48. The equatorial radius of the earth is approximately 3960 mi. Suppose that a wire is wrapped tightly around the earth at the equator. Approximately how much must this wire be lengthened if it is to be strung all the way around the earth on poles 10 ft above the ground? Use the linear approximation formula!
49. The radius of a spherical ball is measured as 10 in., with a maximum error of $\frac{1}{16}$ in. What is the maximum resulting error in its calculated volume?
50. With what accuracy must the radius of the ball of Problem 49 be measured to ensure an error of at most 1 in.^{3} in its calculated volume?

51. The radius of a hemispherical dome is measured as 100 m with a maximum error of 1 cm (Fig. 4.2.16). What is the maximum resulting error in its calculated surface area?

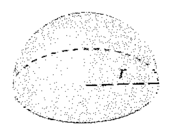

Fig. 4.2.16 The hemisphere of Problem 51: curved surface area $A = 2\pi r^2$

52. With what accuracy must the radius of a hemispherical dome be measured to ensure an error of at most 0.01% in its calculated surface area?

In Problems 53 through 60, a function $f(x)$ and a point $x = a$ are given. Determine graphically an open interval I centered at a so that the function $f(x)$ and its linear approximation $L(x)$ differ by less than the given value ϵ at each point of I.

53. $f(x) = x^2, \quad a = 1, \quad \epsilon = 0.2$
54. $f(x) = \sqrt{x}, \quad a = 1, \quad \epsilon = 0.1$
55. $f(x) = \dfrac{1}{x}, \quad a = 2, \quad \epsilon = 0.01$
56. $f(x) = \sqrt[3]{x}, \quad a = 8, \quad \epsilon = 0.01$
57. $f(x) = \sin x, \quad a = 0, \quad \epsilon = 0.05$
58. $f(x) = \cos x, \quad a = \pi/2, \quad \epsilon = 0.05$
59. $f(x) = \sin x, \quad a = \pi/4, \quad \epsilon = 0.02$
60. $f(x) = \tan x, \quad a = \pi/4, \quad \epsilon = 0.02$

4.3 INCREASING AND DECREASING FUNCTIONS AND THE MEAN VALUE THEOREM

The significance of the *sign* of the first derivative of a function is simple but crucial:

f is increasing on an interval where $f'(x) > 0$;
f is decreasing on an interval where $f'(x) < 0$.

Geometrically, this means that where $f'(x) > 0$, the graph of $y = f(x)$ is rising as you scan it from left to right. Where $f'(x) < 0$, the graph is falling. We can clarify the terms *increasing* and *decreasing* as follows.

Definition ***Increasing and Decreasing Functions***

The function f is **increasing** on the interval $I = (a, b)$ provided that

$$f(x_1) < f(x_2)$$

for all pairs of numbers x_1 and x_2 in I for which $x_1 < x_2$. The function f is **decreasing** on I provided that

$$f(x_1) > f(x_2)$$

for all pairs of numbers x_1 and x_2 for which $x_1 < x_2$.

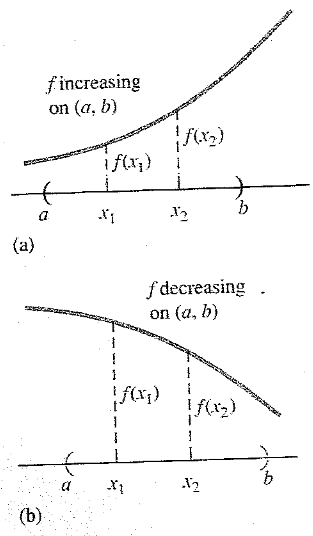

Fig. 4.3.1 (a) An increasing function and (b) a decreasing function

Figure 4.3.1 illustrates this definition. In short, the function f is increasing on $I = (a, b)$ if the values of $f(x)$ increase as x increases (Fig. (4.3.1a)); f is decreasing on I if the values of $f(x)$ decrease as x increases (Fig. (4.3.1b)).

EXAMPLE 1 As illustrated in Fig. 4.3.2, the simple function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, +\infty)$. This follows immediately from the elementary fact that $u^2 < v^2$ if $0 < u < v$. Because $f'(x) = 2x$, we also see immediately that $f'(x) < 0$ on the interval $(-\infty, 0)$ and that $f'(x) > 0$ on the interval $(0, +\infty)$. But for more general functions, the mean value theorem of this section is needed to establish the precise relationship between the sign of the derivative of a function and its increasing-decreasing behavior.