so at maximum altitude, $t = v_0/32 = 10$. At that time, the bolt has reached its maximum altitude of

$$y_{\max} = y(10) = -16 \cdot 10^2 + 320 \cdot 10 = 1600 \quad \text{(ft)}.$$

The result seems contrary to experience. We must conclude that air resistance cannot always be neglected, particularly not in problems involving long journeys at high velocity.

5.2 PROBLEMS

Evaluate the indefinite integrals in Problems 1 through 30.

1. $\int (3x^2 + 2x + 1)\,dx$
2. $\int (3t^4 + 5t - 6)\,dt$
3. $\int (1 - 2x^2 + 3x^3)\,dx$
4. $\int \left(-\frac{1}{t^2}\right) dt$
5. $\int \left(\frac{3}{x^3} + 2x^{3/2} - 1\right) dx$
6. $\int \left(x^{5/2} - \frac{5}{x^4} - \sqrt{x}\right) dx$
7. $\int \left(\frac{3}{2}t^{1/2} + 7\right) dt$
8. $\int \left(\frac{2}{x^{3/4}} - \frac{3}{x^{2/3}}\right) dx$
9. $\int \left(\sqrt[3]{x^2} + \frac{4}{\sqrt[4]{x^5}}\right) dx$
10. $\int \left(2x\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$
11. $\int (4x^3 - 4x + 6)\,dx$
12. $\int \left(\frac{1}{4}t^5 - \frac{5}{t^2}\right) dt$
13. $\int 7\,dx$
14. $\int \left(4\sqrt[3]{x^2} - \frac{6}{\sqrt[3]{x}}\right) dx$
15. $\int (x + 1)^4\,dx$
16. $\int (t + 1)^{10}\,dt$
17. $\int \frac{1}{(x - 10)^7}\,dx$
18. $\int \sqrt{z + 1}\,dz$
19. $\int \sqrt{x}(1 - x)^2\,dx$
20. $\int \sqrt[3]{x}\,(x + 1)^3\,dx$
21. $\int \frac{2x^4 - 3x^3 + 5}{7x^2}\,dx$
22. $\int \frac{(3x + 4)^2}{\sqrt{x}}\,dx$
23. $\int (9t + 11)^5\,dt$
24. $\int \frac{1}{(3z + 10)^7}\,dz$
25. $\int \frac{7}{(x + 77)^2}\,dx$
26. $\int \frac{3}{\sqrt{(x - 1)^3}}\,dx$
27. $\int (5\cos 10x - 10\sin 5x)\,dx$
28. $\int (2\cos \pi x + 3\sin \pi x)\,dx$
29. $\int (3\cos \pi t + \cos 3\pi t)\,dt$
30. $\int (4\sin 2\pi t - 2\sin 4\pi t)\,dt$
31. Verify by differentiation that the integral formulas

$$\int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x + C_1 \quad \text{and}$$

$$\int \sin x \cos x\,dx = -\frac{1}{2}\cos^2 x + C_2$$

are both valid. Reconcile these seemingly different results. What is the relation between the constants C_1 and C_2?

32. Show that the obviously different functions

$$F_1(x) = \frac{1}{1 - x} \quad \text{and} \quad F_2(x) = \frac{x}{1 - x}$$

are both antiderivatives of $f(x) = 1/(1 - x)^2$. What is the relation between $F_1(x)$ and $F_2(x)$?

33. Use the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to find the antiderivatives

$$\int \sin^2 x\,dx \quad \text{and} \quad \int \cos^2 x\,dx.$$

34. (a) First explain why $\int \sec^2 x\,dx = \tan x + C$. (b) Then use the identity $1 + \tan^2 x = \sec^2 x$ to find the antiderivative

$$\int \tan^2 x\,dx.$$

Solve the initial value problems in 35 through 46.

35. $\frac{dy}{dx} = 2x + 1;\ \ y(0) = 3$
36. $\frac{dy}{dx} = (x - 2)^3;\ \ y(2) = 1$
37. $\frac{dy}{dx} = \sqrt{x};\ \ y(4) = 0$
38. $\frac{dy}{dx} = \frac{1}{x^2};\ \ y(1) = 5$
39. $\frac{dy}{dx} = \frac{1}{\sqrt{x + 2}};\ \ y(2) = -1$
40. $\frac{dy}{dx} = \sqrt{x + 9};\ \ y(-4) = 0$
41. $\frac{dy}{dx} = 3x^3 + \frac{2}{x^2};\ \ y(1) = 1$
42. $\frac{dy}{dx} = x^4 - 3x + \frac{3}{x^3};\ \ y(1) = -1$
43. $\frac{dy}{dx} = (x - 1)^3;\ \ y(0) = 2$
44. $\frac{dy}{dx} = \sqrt{x + 5};\ \ y(4) = -3$
45. $\frac{dy}{dx} = \frac{1}{\sqrt{x - 13}};\ \ y(17) = 2$
46. $\frac{dy}{dx} = (2x + 3)^{3/2};\ \ y(3) = 100$

n	$a(P_n)$	$a(Q_n)$
6	2.598076	3.464102
12	3.000000	3.215390
24	3.105829	3.159660
48	3.132629	3.146086
96	3.139350	3.142715
180	3.140955	3.141912
360	3.141433	3.141672
720	3.141553	3.141613
1440	3.141583	3.141598
2880	3.141590	3.141594
5760	3.141592	3.141593

Fig. 5.3.15 Data for estimating π (rounded to six-place accuracy)

$$\underline{A}_n = a(P_n) = n \cdot 2 \cdot \frac{1}{2} \sin \alpha_n \cos \alpha_n = \frac{n}{2} \sin 2\alpha_n = \frac{n}{2} \sin\left(\frac{360^\circ}{n}\right) \tag{19}$$

and that the area of Q_n is

$$\overline{A}_n = a(Q_n) = n \cdot 2 \cdot \frac{1}{2} \tan \alpha_n = n \tan\left(\frac{180^\circ}{n}\right). \tag{20}$$

We substituted selected values of n into Eqs. (19) and (20) to obtain the entries of the table in Fig. 5.3.15. Because $\underline{A}_n \leqq \pi \leqq \overline{A}_n$ for all n, we see that $\pi \approx 3.14159$ to five decimal places. Archimedes' reasoning was *not* circular—he used a direct method for computing the sines and cosines in Eqs. (19) and (20) that does not depend upon a priori knowledge of the value of π. *

5.3 PROBLEMS

Write each of the sums in Problems 1 through 8 in expanded notation.

1. $\sum_{i=1}^{5} 3^i$

2. $\sum_{i=1}^{6} \sqrt{2i}$

3. $\sum_{j=1}^{5} \frac{1}{j+1}$

4. $\sum_{j=1}^{6} (2j-1)$

5. $\sum_{k=1}^{6} \frac{1}{k^2}$

6. $\sum_{k=1}^{6} \frac{(-1)^{k+1}}{k^2}$

7. $\sum_{n=1}^{5} x^n$

8. $\sum_{n=1}^{5} (-1)^{n+1} x^{2n-1}$

Write the sums in Problems 9 through 18 in summation notation.

9. $1 + 4 + 9 + 16 + 25$

10. $1 - 2 + 3 - 4 + 5 - 6$

11. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

12. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$

13. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$

14. $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243}$

15. $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243}$

16. $1 + \sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + \sqrt{6} + \sqrt{7} + 2\sqrt{2} + 3$

17. $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{10}}{10}$

18. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots - \frac{x^{19}}{19}$

Use Eqs. (6) through (9) to find the sums in Problems 19 through 28.

19. $\sum_{i=1}^{10} (4i - 3)$

20. $\sum_{j=1}^{8} (5 - 2j)$

21. $\sum_{i=1}^{10} (3i^2 + 1)$

22. $\sum_{k=1}^{6} (2k - 3k^2)$

23. $\sum_{r=1}^{8} (r-1)(r+2)$

24. $\sum_{i=1}^{5} (i^3 - 3i + 2)$

25. $\sum_{i=1}^{6} (i^3 - i^2)$

26. $\sum_{k=1}^{10} (2k-1)^2$

27. $\sum_{i=1}^{100} i^2$

28. $\sum_{i=1}^{100} i^3$

Use the method of Example 6 to evaluate the limits in Problems 29 and 30.

29. $\lim_{n\to\infty} \frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^3}$

30. $\lim_{n\to\infty} \frac{1^3 + 2^3 + 3^3 + \cdots + n^3}{n^4}$

Use Eqs. (6) through (9) to derive concise formulas in terms of n for the sums in Problems 31 and 32.

31. $\sum_{i=1}^{n} (2i-1)$

32. $\sum_{i=1}^{n} (2i-1)^2$

In Problems 33 through 42, let R denote the region that lies below the graph of $y = f(x)$ over the interval $[a, b]$ on the x-axis. Use the method of Example 1 to calculate both an underestimate $\underline{A}_n$ and an overestimate $\overline{A}_n$ for the area A of R, based on a division of $[a, b]$ into n subintervals all with the same length $\Delta x = (b-a)/n$.

33. $f(x) = x$ on $[0, 1]$; $n = 5$

34. $f(x) = x$ on $[1, 3]$; $n = 5$

35. $f(x) = 2x + 3$ on $[0, 3]$; $n = 6$

36. $f(x) = 13 - 3x$ on $[0, 3]$; $n = 6$ (Fig. 5.3.16)

37. $f(x) = x^2$ on $[0, 1]$; $n = 5$

*See Chapter 2 of C. H. Edwards, Jr., *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979).