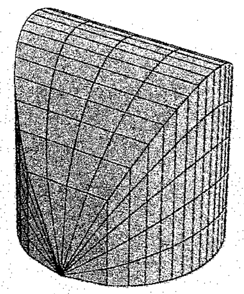

Fig. 6.2.33 The observatory of Problem 39

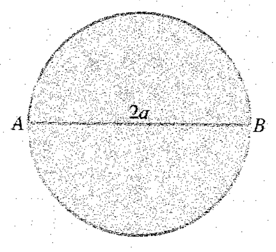

Fig. 6.2.34 The circular base of the observatory (Problem 39)

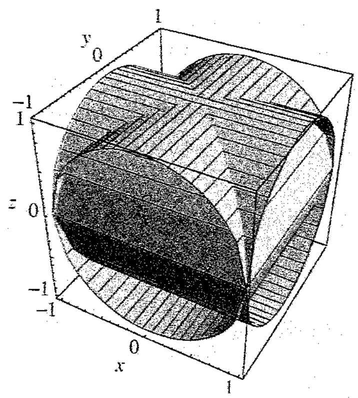

Fig. 6.2.37 The intersecting cylinders of Problem 47

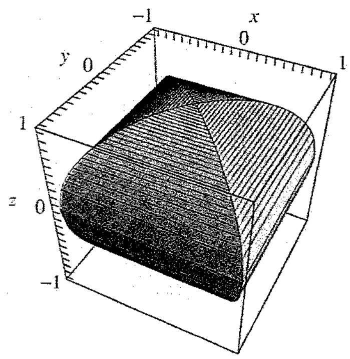

Fig. 6.2.38 The solid of intersection (Problem 47)

42. The base of a solid is the region in the xy-plane bounded by the parabolas $y = x^2$ and $x = y^2$. Find the volume of this solid if every cross section perpendicular to the x-axis is a square with *its* base in the xy-plane.

43. The *paraboloid* generated by rotating around the x-axis the region under the parabola $y^2 = 2px$, $0 \leqq x \leqq h$, is shown in Fig. 6.2.35. Show that the volume of the paraboloid is one-half that of the circumscribed cylinder also shown in the figure.

44. A pyramid has height h and rectangular base with area A. Show that its volume is $V = \frac{1}{3}Ah$. (*Suggestion:* Note that each cross section parallel to the base is a rectangle.)

45. Repeat Problem 44, except make the base a triangle with area A.

46. Find the volume that remains after a hole of radius 3 is bored through the center of a solid sphere of radius 5 (Fig. 6.2.36).

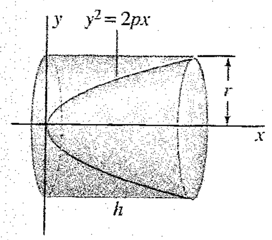

Fig. 6.2.35 The paraboloid and cylinder of Problem 43

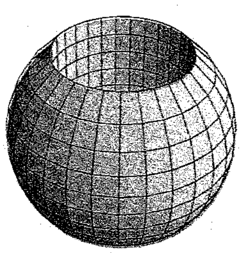

Fig. 6.2.36 The sphere-with-hole of Problem 46

47. Two horizontal circular cylinders both have radius a, and their axes intersect at right angles. Find the volume of their solid of intersection (Figs. 6.2.37 and 6.2.38, where $a = 1$). Is it clear to you that each horizontal cross section of the solid is a square?

48. Figure 6.2.39 shows a "spherical segment" of height h that is cut off from a sphere of radius r by a horizontal plane. Show that its volume is

$$V = \tfrac{1}{3}\pi h^2(3r - h).$$

49. A doughnut-shaped solid, called a *torus* (Fig. 6.2.40), is generated by revolving around the y-axis the circular disk

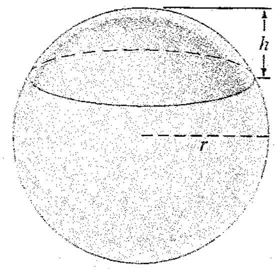

Fig. 6.2.39 A spherical segment (Problem 48)

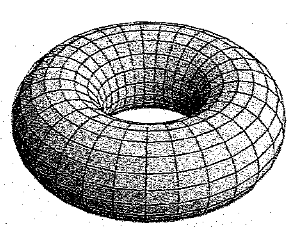

Fig. 6.2.40 The torus of Problem 49

$(x - b)^2 + y^2 \leqq a^2$ centered at the point $(b, 0)$, where $0 < a < b$. Show that the volume of this torus is $V = 2\pi^2 a^2 b$. (*Suggestion:* Note that each cross section perpendicular to the y-axis is an annular ring, and recall that

$$\int_0^a \sqrt{a^2 - y^2}\, dy = \tfrac{1}{4}\pi a^2$$

because the integral represents the area of a quarter-circle of radius a.)

50. The summit of a hill is 100 ft higher than the surrounding level terrain, and each horizontal cross section of the hill is circular. The following table gives the radius r (in feet) for selected values of the height h (in feet) above the surrounding terrain. Use Simpson's approximation to estimate the volume of the hill.

h	0	25	50	75	100
r	60	55	50	35	0

51. *Newton's Wine Barrel* Consider a barrel with the shape of the solid generated by revolving around the x-axis the region under the parabola

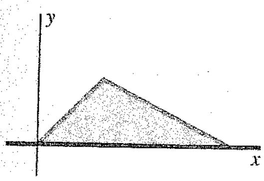

Fig. 6.3.17 Problem 7

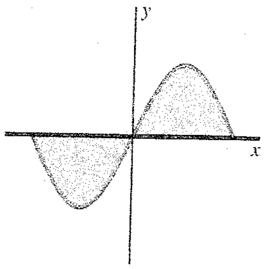

Fig. 6.3.18 Problem 11

9. $y = 2x^2, y^2 = 4x$; the x-axis
10. $y = 3x - x^2, y = 0$; the y-axis
11. $y = 4x - x^3, y = 0$; the y-axis (Fig. 6.3.18)
12. $x = y^3 - y^4, x = 0$; the line $y = -2$ (Fig. 6.3.19)
13. $y = x - x^3, y = 0$ $(0 \leqq x \leqq 1)$; the y-axis
14. $x = 16 - y^2, x = 0, y = 0$ $(0 \leqq y \leqq 4)$; the x-axis
15. $y = x - x^3, y = 0$ $(0 \leqq x \leqq 1)$; the line $x = 2$ (Fig. 6.3.20)
16. $y = x^3, y = 0, x = 2$; the y-axis (Fig. 6.3.21)
17. $y = x^3, y = 0, x = 2$; the line $x = 3$
18. $y = x^3, y = 0, x = 2$; the x-axis
19. $y = x^2, y = 0, x = -1, x = 1$; the line $x = 2$
20. $y = x^2, y = x$ $(0 \leqq x \leqq 1)$; the y-axis
21. $y = x^2, y = x$ $(0 \leqq x \leqq 1)$; the x-axis
22. $y = x^2, y = x$ $(0 \leqq x \leqq 1)$; the line $y = 2$
23. $y = x^2, y = x$ $(0 \leqq x \leqq 1)$; the line $x = -1$
24. $x = y^2, x = 2 - y^2$; the x-axis (Fig. 6.3.22)

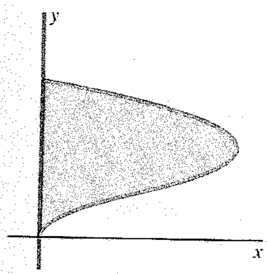

Fig. 6.3.19 Problem 12

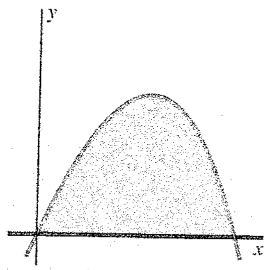

Fig. 6.3.20 Problem 15

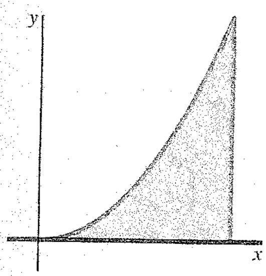

Fig. 6.3.21 Problem 16

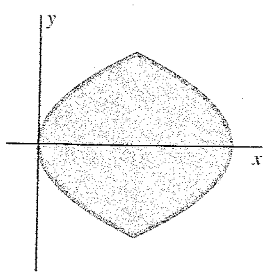

Fig. 6.3.22 Problem 24

25. $x = y^2, x = 2 - y^2$; the line $y = 1$
26. $y = 4x - x^2, y = 0$; the y-axis
27. $y = 4x - x^2, y = 0$; the line $x = -1$
28. $y = x^2, x = y^2$; the line $y = -1$

In Problems 29 through 34, first use a calculator or computer to approximate (graphically or otherwise) the points of intersection of the two given curves. Let R be the region bounded by these curves. Integrate to approximate the volume of the solid obtained by revolving the region R around the y-axis. In Problems 31 through 34 you will find helpful the integral formula

$$\int u \cos u \, du = \cos u + u \sin u + C,$$

which you can verify by differentiation of the right-hand side.

29. $y = x^3 + 1$, $y = 6x - x^2$ (R lies to the right of the y-axis)
30. $y = x^4$, $y = 10x - 5$
31. $y = \cos x$, $y = x^2$
32. $y = \cos x$, $y = (x - 1)^2$
33. $y = \cos x$, $y = 3x^2 - 6x + 2$
34. $y = 3 \cos x$, $y = -\cos 4x$ (R lies between $x = -2$ and $x = 2$)

35. Verify the formula for the volume of a right circular cone by using the method of cylindrical shells. Apply the method to the figure generated by rotating the triangular region with vertices $(0, 0)$, $(r, 0)$, and $(0, h)$ around the y-axis.

36. Use the method of cylindrical shells to compute the volume of the paraboloid of Problem 43 in Section 6.2.

37. Use the method of cylindrical shells to find the volume of the ellipsoid obtained by revolving the elliptical region bounded by the graph of the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

around the y-axis.

38. Use the method of cylindrical shells to derive the formula given in Problem 48 of Section 6.2 for the volume of a spherical segment.

39. Use the method of cylindrical shells to compute the volume of the torus in Problem 49 in Section 6.2. [*Suggestion:* Substitute u for $x - b$ in the integral given by the formula in Eq. (2).]

40. (a) Find the volume of the solid generated by revolving the region bounded by the curves $y = x^2$ and $y = x + 2$ around the line $x = -2$. (b) Repeat part (a), but revolve the region around the line $x = 3$.

41. Find the volume of the solid generated by revolving the circular disk $x^2 + y^2 \leqq a^2$ around the vertical line $x = a$.

42. (a) Verify by differentiation that

$$\int x \sin x \, dx = \sin x - x \cos x + C.$$

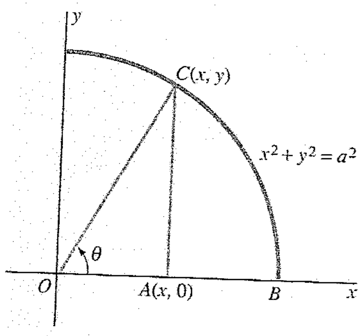

Fig. 9.6.8 The circular sector of Problem 43

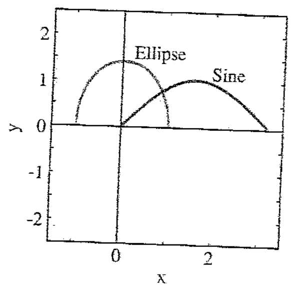

Fig. 9.6.9 Two arcs with the same length (Problem 46)

48. Compute the area of the surface obtained by revolving around the y-axis the curve of Problem 47.

49. A torus (see Fig. 9.6.10) is obtained by revolving around the y-axis the circle

$$(x - b)^2 + y^2 = a^2 \qquad (0 < a \leqq b).$$

Show that the surface area of the torus is $4\pi^2 ab$.

50. Find the area under the curve $y = \sqrt{9 + x^2}$ over the interval $[0, 4]$.

51. Find the area of the surface obtained by revolving around the x-axis the curve $y = \sin x, 0 \leqq x \leqq \pi$ (see Fig. 9.6.11).

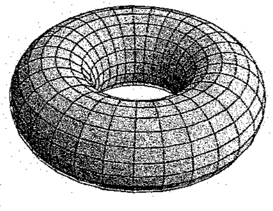

Fig. 9.6.10 The torus of Problem 49

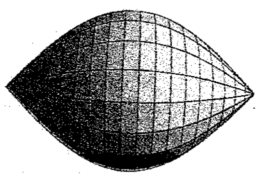

Fig. 9.6.11 The pointed football of Problem 51

52. An ellipsoid of revolution is obtained by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$ around the x-axis. Suppose that $a > b$. Show that the ellipsoid has surface area

$$A = 2\pi ab\left[\frac{b}{a} + \frac{a}{c}\sin^{-1}\left(\frac{c}{a}\right)\right],$$

where $c = \sqrt{a^2 - b^2}$. Assume that $a \approx b$, so that $c \approx 0$ and $\sin^{-1}(c/a) \approx c/a$. Conclude that $A \approx 4\pi a^2$.

53. Suppose that $b > a$ for the ellipsoid of revolution of Problem 52. Show that its surface area is then

$$A = 2\pi ab\left[\frac{b}{a} + \frac{a}{c}\ln\left(\frac{b + c}{a}\right)\right],$$

where $c = \sqrt{b^2 - a^2}$. Use the fact that $\ln(1 + x) \approx x$ if $x \approx 0$, and thereby conclude that $A \approx 4\pi a^2$ if $a \approx b$.

54. A road is to be built from the point $(2, 1)$ to the point $(5, 3)$, following the path of the parabola

$$y = -1 + 2\sqrt{x - 1}.$$

Calculate the length of this road (the units on the coordinate axes are in miles). (*Suggestion:* Substitute $x = \sec^2 \theta$ into the arc-length integral.)

55. Suppose that the cost of the road in Problem 54 is $\sqrt{x}$ million dollars per mile. Calculate the total cost of the road.

56. A kite is flying at a height of 500 ft and at a horizontal distance of 100 ft from the string-holder on the ground. The kite string weighs 1/16 oz/ft and is hanging in the shape of the parabola $y = x^2/20$ that joins the string-holder at $(0, 0)$ to the kite at $(100, 500)$ (Fig. 9.6.12). Calculate the work (in foot-pounds) done in lifting the kite string from the ground to its present position.

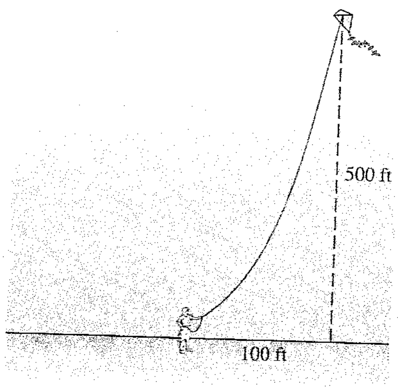

Fig. 9.6.12 The kite string of Problem 56

9.7 INTEGRALS CONTAINING QUADRATIC POLYNOMIALS

Many integrals containing a square root or negative power of a quadratic polynomial $ax^2 + bx + c$ can be simplified by the process of *completing the square*. For example,

$$x^2 + 2x + 2 = (x + 1)^2 + 1,$$

and hence the substitution $u = x + 1, du = dx$ yields

$$\int \frac{1}{x^2 + 2x + 2}\,dx = \int \frac{1}{u^2 + 1}\,du = \tan^{-1} u + C = \tan^{-1}(x + 1) + C.$$

9. $\displaystyle\int_0^9 \frac{1}{(9-x)^{3/2}}\,dx$

10. $\displaystyle\int_0^3 \frac{1}{(x-3)^2}\,dx$

11. $\displaystyle\int_{-\infty}^{-2} \frac{1}{(x+1)^3}\,dx$

12. $\displaystyle\int_{-\infty}^{0} \frac{1}{\sqrt{4-x}}\,dx$

13. $\displaystyle\int_{-1}^{8} \frac{1}{\sqrt[3]{x}}\,dx$

14. $\displaystyle\int_{-4}^{4} \frac{1}{(x+4)^{2/3}}\,dx$

15. $\displaystyle\int_{2}^{\infty} \frac{1}{\sqrt[3]{x-1}}\,dx$

16. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+4)^{3/2}}\,dx$

17. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+4}\,dx$

18. $\displaystyle\int_{0}^{\infty} e^{-(x+1)}\,dx$

19. $\displaystyle\int_{0}^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

20. $\displaystyle\int_{0}^{2} \frac{x}{x^2-1}\,dx$

21. $\displaystyle\int_{0}^{\infty} xe^{-3x}\,dx$

22. $\displaystyle\int_{-\infty}^{2} e^{2x}\,dx$

23. $\displaystyle\int_{0}^{\infty} xe^{-x^2}\,dx$

24. $\displaystyle\int_{-\infty}^{\infty} |x|e^{-x^2}\,dx$

25. $\displaystyle\int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

26. $\displaystyle\int_{0}^{\infty} \frac{x}{1+x^2}\,dx$

27. $\displaystyle\int_{0}^{\infty} \cos x\,dx$

28. $\displaystyle\int_{0}^{\infty} \sin^2 x\,dx$

29. $\displaystyle\int_{1}^{\infty} \frac{\ln x}{x}\,dx$

30. $\displaystyle\int_{2}^{\infty} \frac{1}{x\ln x}\,dx$

31. $\displaystyle\int_{2}^{\infty} \frac{1}{x(\ln x)^2}\,dx$

32. $\displaystyle\int_{1}^{\infty} \frac{\ln x}{x^2}\,dx$

33. $\displaystyle\int_{0}^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx$

34. $\displaystyle\int_{0}^{\pi/2} \frac{\sin x}{(\cos x)^{4/3}}\,dx$

35. $\displaystyle\int_{0}^{1} \ln x\,dx$

36. $\displaystyle\int_{0}^{1} \frac{\ln x}{x}\,dx$

37. $\displaystyle\int_{0}^{1} \frac{\ln x}{x^2}\,dx$

38. $\displaystyle\int_{0}^{\infty} e^{-x}\cos x\,dx$

In Problems 39 through 42, the given integral is improper both because the interval of integration is unbounded and because the integrand is unbounded near zero. Investigate its convergence by expressing it as a sum of two integrals—one from 0 to 1, the other from 1 to ∞. Evaluate those integrals that converge.

39. $\displaystyle\int_{0}^{\infty} \frac{1}{x+x^2}\,dx$

40. $\displaystyle\int_{0}^{\infty} \frac{1}{x^2+x^4}\,dx$

41. $\displaystyle\int_{0}^{\infty} \frac{1}{x^{1/2}+x^{3/2}}\,dx$

42. $\displaystyle\int_{0}^{\infty} \frac{1}{x^{2/3}+x^{4/3}}\,dx$

In Problems 43 through 46, find all real number values of k for which the given improper integral converges. Evaluate the integral for those values of k.

43. $\displaystyle\int_{0}^{1} \frac{1}{x^k}\,dx$

44. $\displaystyle\int_{1}^{\infty} \frac{1}{x^k}\,dx$

45. $\displaystyle\int_{0}^{1} x^k\ln x\,dx$

46. $\displaystyle\int_{1}^{\infty} \frac{1}{x(\ln x)^k}\,dx$

47. Beginning with the definition of the gamma function in Eq. (7), integrate by parts to show that

$$\Gamma(x+1) = x\Gamma(x)$$

for every positive real number x.

48. Explain how to apply the result of Problem 47 n times in succession to show that if n is a positive integer, then $\Gamma(n+1) = n!\Gamma(1) = n!$.

Problems 49 through 51 deal with Gabriel's horn, the surface obtained by revolving the curve $y = 1/x$, $x \geqq 1$, around the x-axis (Fig. 9.8.12).

49. Show that the area under the curve $y = 1/x$, $x \geqq 1$, is infinite.

50. Show that the volume of revolution enclosed by Gabriel's horn is finite, and compute it.

51. Show that the surface area of Gabriel's horn is infinite. (*Suggestion:* Let A_t denote the surface area from $x = 1$ to $x = t > 1$. Prove that $A_t > 2\pi \ln t$.) In any case, the implication is that we could fill Gabriel's horn with a finite amount of paint (Problem 50), but no finite amount suffices to paint its surface.

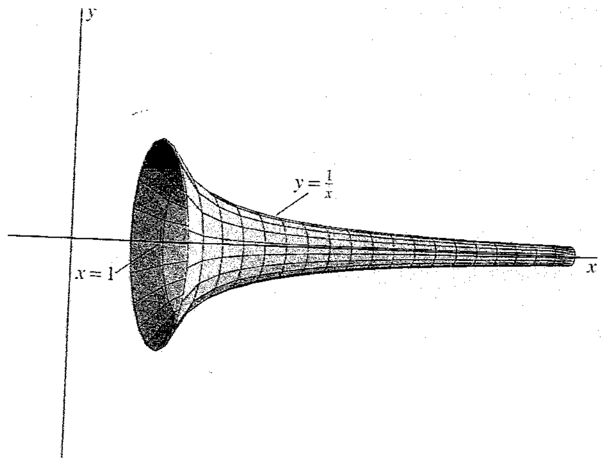

Fig. 9.8.12 Gabriel's horn (Problems 49 through 51)

52. Show that

$$\int_{-\infty}^{\infty} \frac{1+x}{1+x^2}\,dx$$

diverges, but that

$$\lim_{t\to\infty} \int_{-t}^{t} \frac{1+x}{1+x^2}\,dx = \pi.$$

53. Use the substitution $x = e^{-u}$ and the fact that $\Gamma(n+1) = n!$ (Problem 48) to prove that if m and n are fixed but arbitrary positive integers, then