1. Compute the limits

(a) $\lim\limits_{n\to\infty} \frac{\sin n}{n}$

(b) $\lim\limits_{x\to 0} \frac{1-\cos x}{x}$

(c) $\lim\limits_{h\to 0} \frac{\sqrt{1+h}-1}{h}$

(5+5+5)

(a) $\left|\frac{\sin n}{n}\right| \leq \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

so $\lim\limits_{n\to\infty} \frac{\sin n}{n} = 0$ by the "squeeze law".

(b) $$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\left(1-\frac{1}{2}x^2+O(x^4)\right)}{x}$$

$$= \lim_{x\to 0}\left(\frac{1}{2}x + O(x^3)\right) = 0$$

(c) $$\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \lim_{h\to 0} \frac{\left(\sqrt{1+h}-1\right)\left(\sqrt{1+h}+1\right)}{h\left(\sqrt{1+h}+1\right)} = \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1} = \frac{1}{2}$$

Or: with $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$$\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = f'(1) = \frac{1}{2}.$$

2. Prove that the equation

$$\underbrace{x^3 - 4x + 1}_{=: f(x)} = 0$$

has three solutions.

Hint: Don't try to compute the solutions! (10)

$$\lim_{x \to \infty} f(x) = \infty$$

$$f(1) = -2$$

$$f(0) = 1$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

Since f is continuous, the graph of f must cross the x-axis at least 3 times (intermediate value theorem), so we have at least 3 solutions.

A cubic equation generally has at most three real solutions, so there must be exactly three solutions to $f(x) = 0$.

3. Consider the function

$$f(x) = \frac{\ln x}{x}.$$

What is the domain of f? Find horizontal and vertical asymptotes, local minima, local maxima, and inflection points of f. Identify the regions where the graph of f is concave upward or concave downward. Finally, sketch the graph into the coordinate system provided. (15)

$\mathcal{D}(f) = \mathbb{R}_+$, the positive real numbers.

$\lim\limits_{x \to \infty} f(x) = 0$, $\lim\limits_{x \to 0} f(x) = -\infty$ by properties of the logarithm.

$$f'(x) = \frac{1}{x^2} - \ln x \frac{1}{x^2} = (1 - \ln x)\frac{1}{x^2}$$

$$f''(x) = -\frac{1}{x^3} + (1 - \ln x)(-2)\frac{1}{x^3} = (-3 + 2\ln x)\frac{1}{x^3}$$

critical points: $f'(x) = 0$ if $\ln x = 1 \Rightarrow x = e$

clearly, f' changes sign from + to - there,

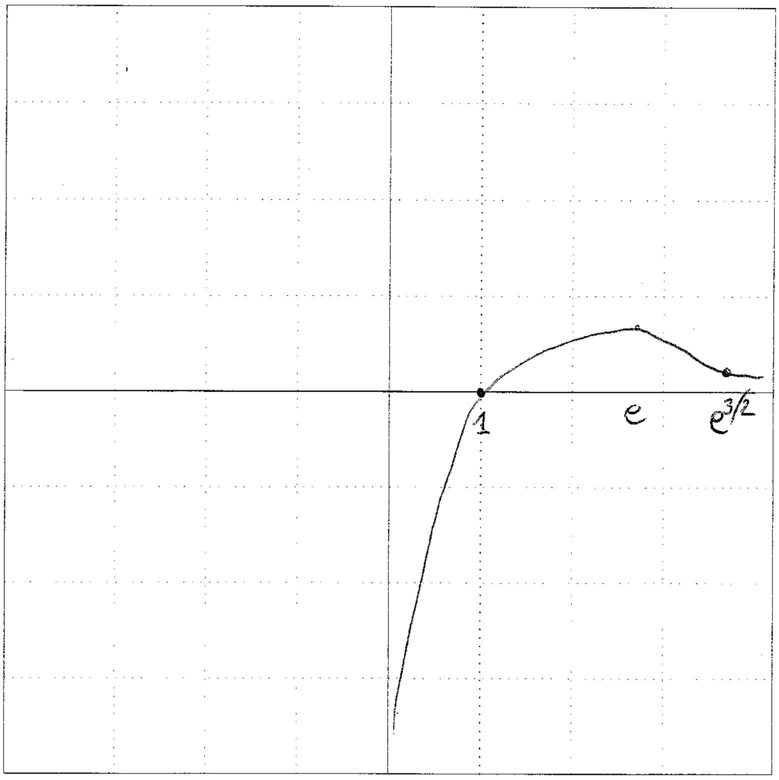
so we have a maximum at (e, e^{-1})

points of inflection: $f''(x) = 0$ if $3 = 2\ln x \Rightarrow x = e^{\frac{3}{2}}$

clearly, f'' changes sign from - to + there,

so graph changes from concave down to concave up;

point of inflection at $(e^{\frac{3}{2}}, \frac{3}{2} e^{-\frac{3}{2}})$


1
e
$e^{3/2}$

4. Compute the indefinite integrals

(a) $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

(b) $\int \frac{4x-2}{x^3-x}\,dx$

(10+10)

(a) Trigonometric substitution $x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$

$$\Rightarrow \int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4\sin^2\theta \;\, 2\cos\theta\, d\theta}{\sqrt{4-4\sin^2\theta}}$$

$$= 4\int \sin^2\theta\, d\theta = 2\int (1-\cos 2\theta)\, d\theta$$

$$= 2\theta - \sin 2\theta + C_1 = 2\arcsin\frac{x}{2} - \sin\left(2\arcsin\frac{x}{2}\right) + C_1$$

or better: $\quad \ldots = 2\theta - 2\sin\theta\cos\theta + C$

$$= 2\arcsin\frac{\theta}{2} - x\sqrt{1-\left(\frac{x}{2}\right)^2} + C$$

(b) Partial fractions: $\frac{4x-2}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} = \frac{A(x^2-1) + Bx(x-1) + Cx(x+1)}{x(x+1)(x-1)}$

so: $\left.\begin{aligned} A+B+C &= 0 \\ -B+C &= 4 \\ A &= 2 \end{aligned}\right\}$ $\quad 2+2C = 4 \Rightarrow C = 1 \Rightarrow B = -3$

$$\Rightarrow \int \frac{4x-2}{x^3-x}\,dx = 2\int\frac{dx}{x} - 3\int\frac{dx}{x+1} + \int\frac{dx}{x-1}$$

$$= 2\ln|x| - 3\ln|x+1| + \ln|x-1| + C$$

5. Compute the improper integrals

(a) $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

(b) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(\cos x)^{4/3}}\,dx$

(10+10)

(a) let $u = \sqrt{x} \Rightarrow du = \frac{1}{2} x^{-\frac{1}{2}} dx$

$$\Rightarrow \int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int_0^1 e^u\,du = 2e^u\Big|_0^1 = 2(e-1)$$

(b) let $u = \cos x \Rightarrow du = -\sin x\,dx$

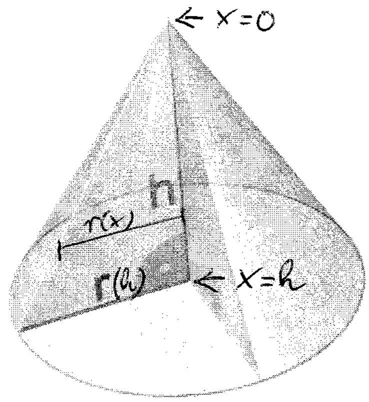
$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{(\cos x)^{4/3}}\,dx = -\int_1^0 \frac{du}{u^{4/3}}$$

$$= 3u^{-\frac{1}{3}}\Big|_1^0 = \infty$$

so the integral diverges.

6. Show, using integration, that the volume of a right circular cone of height h and radius r (see Figure) is given by (10)

$$V = \frac{1}{3}\pi h r^2.$$



Best use method of cross-sections:

$$V = \int_0^h A(x)\,dx$$

where $A(x)$ is the area of the cross-section at height x, so

$$A(x) = \pi r^2(x) = \pi \left(r\frac{x}{h}\right)^2$$

$$\Rightarrow V = \pi \frac{r^2}{h^2} \underbrace{\int_0^h x^2\,dx}_{=\frac{1}{3}h^3} = \frac{1}{3}\pi r^2 h.$$

7. Compute the Taylor series about $x = 0$ of

(a) $f(x) = \dfrac{1}{1-x}$

(b) $f(x) = \dfrac{1-x^2}{1-x}$

(c) $f(x) = \dfrac{1-x^3}{1-x}$

(d) Do you see a pattern? Can you formulate a more general statement?

(5+5+5+5)

This question can be solved entirely by noting the formula for the finite geometric series, namely

$$\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$$

as proved in class. So, for $N=1$,

$$1+x = \frac{1-x^2}{1-x} \longrightarrow \text{part (b)}$$

for $N=2$,

$$1+x+x^2 = \frac{1-x^3}{1-x} \longrightarrow \text{part (c)}$$

for $|x|<1$ and $N \to \infty$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \longrightarrow \text{part (a)}$$

If you did not see the geometric series straight away, you'd have to compute:

(a) $f(x) = (1-x)^{-1}$

$f'(x) = (1-x)^{-2}$

$f''(x) = 2(1-x)^{-3}$

$\vdots$

$f^{(n)}(x) = n!\,(1-x)^{-(n+1)} \Rightarrow f^{(n)}(0) = n!$

So $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n$

(b) $f(x) = \frac{1-x^2}{1-x} = \frac{(1-x)(1+x)}{1-x} = 1+x$

So f is a polynomial where it is defined (i.e. for $x \neq 1$), and the Taylor series of a polynomial is the same polynomial

(c) Use long division:

$$
\begin{array}{rl}
(-x^3+1)/(-x+1) & = x^2+x+1 \\
-)\,-x^3+x^2 & \\
\hline
-x^2 & \\
-)\,-x^2+x & \\
\hline
-x+1 & \\
-)\,-x+1 & \\
\hline
0 &
\end{array}
$$

$\Rightarrow f(x) = x^2+x+1$ (except at $x=1$) and is hence its own Taylor series.

8. For which values of p does the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

converge, respectively diverge? (10)

Integral test: let $u = \ln n \Rightarrow du = \frac{dn}{n}$

$$\int_2^{\infty} \frac{1}{n(\ln n)^p}\, dn = \int_{\ln 2}^{\infty} \frac{du}{u^p}$$

$$= \begin{cases} \ln u \Big|_{\ln 2}^{\infty} = \infty & \text{when } p = 1 \\ \frac{1}{1-p} u^{1-p} \Big|_{\ln 2}^{\infty} & \text{when } p \neq 1 \end{cases}$$

Thus, the integral converges only if $p > 1$.

9. Let $P(t)$ denote the number of fish in a lake at time t, and let C denote the "carrying capacity" of the lake. Suppose further that fishermen catch a fraction k of the fish per unit of time, so that the population satisfies the equation

$$\frac{dP}{dt} = \left(1 - \frac{P}{C}\right) P - kP \qquad \text{with} \qquad P(t) = P_0 . \qquad \text{so } 0 < k < 1$$

(a) For given values of C and k, when is the population increasing, when is it decreasing?

(b) For given values of C and k, how many fish will be in the lake in the long run? (You do not need to solve the differential equation to answer this question!)

(c) Which value of the fishing rate k maximizes the number of fish caught?

(d) Solve the differential equation explicitly and check that your solution is consistent with your answer to part (b).

(5+5+10+10)

Note: $\frac{dP}{dt} = \left(1 - k - \frac{P}{C}\right) P$

(a) $\frac{dP}{dt} > 0$ if $1 - k - \frac{P}{C} > 0$, i.e., $P < C(1-k)$

similarly, $\frac{dP}{dt} < 0$ if $P > C(1-k)$

(b) Since the population is increasing for $P < C(1-k)$ and decreasing for $P > C(1-k)$, we expect that $P \to C(1-k)$ as $t \to \infty$. (A rigorous justification requires more than this heuristics, but we shall see that the result is ideed correct from the full solution in part (d). When a full solution is not available, a proof can be achieved via "linear stability analysis", but that's for an introductory course on differential equations.)

(c) The fishing yield in the long run will be

$$Y = kP = k\,C(1-k) = C(k-k^2)$$

Now find maximum with respect to k:

$$\frac{dY}{dk} = C(1-2k)$$

$$\frac{dY}{dk} = 0 \quad \Rightarrow \quad k = \frac{1}{2}$$

(Since $Y(k)$ is an upside-down parabola, this critical point must correspond to a maximum.)

So the optimal fishing rate constant is $k = \frac{1}{2}$.

(d) $$\int_{P_0}^{P(t)} \frac{dP}{\left(1-k-\frac{P}{C}\right)P} = \int_0^t dt$$

need partial fractions: $$\frac{A}{C(1-k)-P} + \frac{B}{P} = \frac{AP + B\left(C(1-k)-P\right)}{\left(C(1-k)-P\right)P}$$

$$\Rightarrow \quad A - B = 0\,, \quad B\,C(1-k) = 1 \quad \Rightarrow B = \frac{1}{C(1-k)}$$

$$\Rightarrow \int_{P_0}^{P(t)} \frac{dP}{\left(1-k-\frac{P}{C}\right)P} = \frac{1}{1-k}\int_{P_0}^{P(t)}\left(\frac{1}{C(1-k)-P} + \frac{1}{P}\right)dP = \frac{1}{1-k}\ln\frac{P}{C(1-k)-P}\Bigg|_{P_0}^{P(t)} = t$$

$$\Rightarrow \ln \frac{P(t)}{C(1-k)-P(t)} - \ln \frac{P_0}{C(1-k)-P_0} = (1-k)t$$

$$\Rightarrow \frac{P(t)}{C(1-k)-P(t)} = \frac{P_0}{C(1-k)-P_0} e^{(1-k)t}$$

now solve for $P(t)$:

$$P(t) = \frac{c(1-k)\, P_0 e^{(1-k)t}}{c(1-k) - P_0 + P_0 e^{(1-k)t}}$$

$$= \frac{C(1-k)\, P_0}{P_0 + \left(c(1-k) - P_0\right) e^{(k-1)t}}$$

$$\longrightarrow \frac{c(1-k)\, P_0}{P_0} = C(1-k) \qquad \text{as } t \to \infty$$

so this is consistent with (b).