Derivatives Lab

Session 17

November 15, 2011

1. Consider the Black–Scholes partial differential equation for the price $C(S,t)$ of a European option as function of the current stock price S and time t,
$$\frac{\partial C}{\partial t} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 C}{\partial S^2} + r\,S\,\frac{\partial C}{\partial S} - r\,C = 0\,,$$
where σ is the volatility of the underlying stock and r the risk-free interest rate.

   Use the chain rule to show that, under the change of variable $S = \exp(X)$ and $V(X,t) = C(S,t)$, the Black–Scholes equation turns into the constant coefficient drift-diffusion equation
$$\frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2\,\frac{\partial^2 V}{\partial X^2} + \left(r - \frac{1}{2}\,\sigma^2\right)\frac{\partial V}{\partial X} - r\,V = 0\,.$$

2. A system of linear equation of the form
$$\begin{pmatrix} b_1 & c_1 & & \cdots & 0 \\ a_1 & b_2 & c_2 & & \\ & a_2 & b_3 & \ddots & \\ \vdots & & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & & a_{n-1} & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{pmatrix}$$
where the left hand $n \times n$ matrix is *tridiagonal*, i.e., is zero except for the three main diagonals, can be easily solved in $O(n)$ steps.

   (a) Write out the expressions which arise when performing Gaussian elimination on this system.

   (b) Write a tridiagonal solver as a Python function.

   (c) Scipy has a build-in banded matrix solver:

   ```
   from scipy.linalg import solve_banded
   ```

   Look up the documentation and use it to compare results and compute time against your tridiagonal solver for the case when $a_i = c_i = 1$ for $i = 1, \ldots, n-1$ and $b_i = -2$ for $i = 1, \ldots, n$ when n is large and the right hand side some random vector.