Jacobs University
School of Engineering and Science
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Fall Term 2011

Homework Set 1

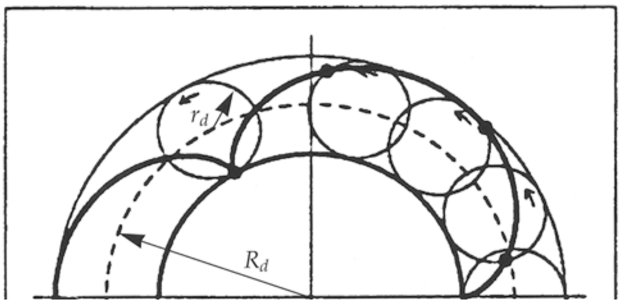
Perspectives of Mathematics

Homework Problems: The Feigenbaum Diagram

1.1. *Attracting Orbits of Periods* 1 *and* 2 *for the Mandelbrot Set.*

a) Consider quadratic polynomials $p_c(z) = z^d + c$ with a complex parameter c, for a degree $d \geq 2$. Determine the set C_d of parameters c for which p_c has an attracting fixed point. (*Specifically for* $d = 2$, *this is* $\{c \in \mathbb{C} \colon c = \lambda/2 - (\lambda/2)^2\}$ *with* $\lambda \in \mathbb{C}$, $|\lambda| < 1|$).

Show that the boundary of C_d is a *cardioid*: set $R_d := d^{-1/(d-1)}$ and $r_d = R_d/d$. Draw a circle C with radius $R_d - r_d$ around the origin and consider a small disk D of radius r_d that touches C from the outside on the right (so that the center of the disk is at the point R_d). Attach a pen to D to the point where it touches C. Now let D roll along the boundary of C: then the pen draws exactly the cardioid C_d.



b) Specifically for $d = 2$, show that p_c has an attracting orbit of period 2 if and only if $|c - 1| < 1/4$, i.e., c is in a perfect disk around -1 with radius $1/4$.

1.2. *Number of Periodic Points.*

a) Find the number of periodic points of period n that a quadratic polynomial has, for $n = 1, 2, 3, \ldots, 11$ (counting multiplicities). Show that this number equals $2^n - 2$ if and only if n is prime.

b) Do the same for a cubic polynomial.

c) Specifically for $z \mapsto z^2$, find all periodic points of period 1, 2, 3, and 4. Do the same for $z \mapsto z^2 - 2$ and for $z \mapsto z^2 - 0.75$. In the latter case, explain that you found all periodic points.

1.3. *Conjugation of Polynomials and Newton Maps.*

a) For the logistic family $f_\mu(x) = \mu x(1 - x)$ and the Mandelbrot family $p_c(z) = z^2 + c$, find out which f_μ is conjugate to which p_c by a map $z = ax + b$. Explain how you find the Feigenbaum diagram within the Mandelbrot set.

b) Show that every cubic polynomial can be conjugated to a polynomial $z^3 + az + b$. Is this polynomial unique?

c) For a polynomial p, the associated Newton map is $N_p(z) = z - p(z)/p'(z)$. Show that all quadratic polynomials have conjugated Newton maps. Show that all cubic polynomials p maps have their Newton maps conjugate to the Newton map for $q_\lambda(z) = z(z - 1)(z - \lambda)$ for some $\lambda \in \mathbb{C}$. Is that λ unique?

Due Date: Wednesday, 19 October 2011, at the beginning of class.

You may work in groups of up to two people, but both of you should submit your own solutions.