1. The figure below[1] is one of the most cited and reproduced graphs in atmospheric science. It shows aircraft measurements of the energy content ("Spectral Density") E vs. the size, expressed in terms of the wave number k, of turbulent structures in the atmosphere.

   The authors claim to have discovered a certain type of relation for E as a function of k, which they indicate in their figure by straight lines. What is their claim? (Note that they only care about the slopes of the lines, so you need not worry about measuring the y-intercepts!) (15)

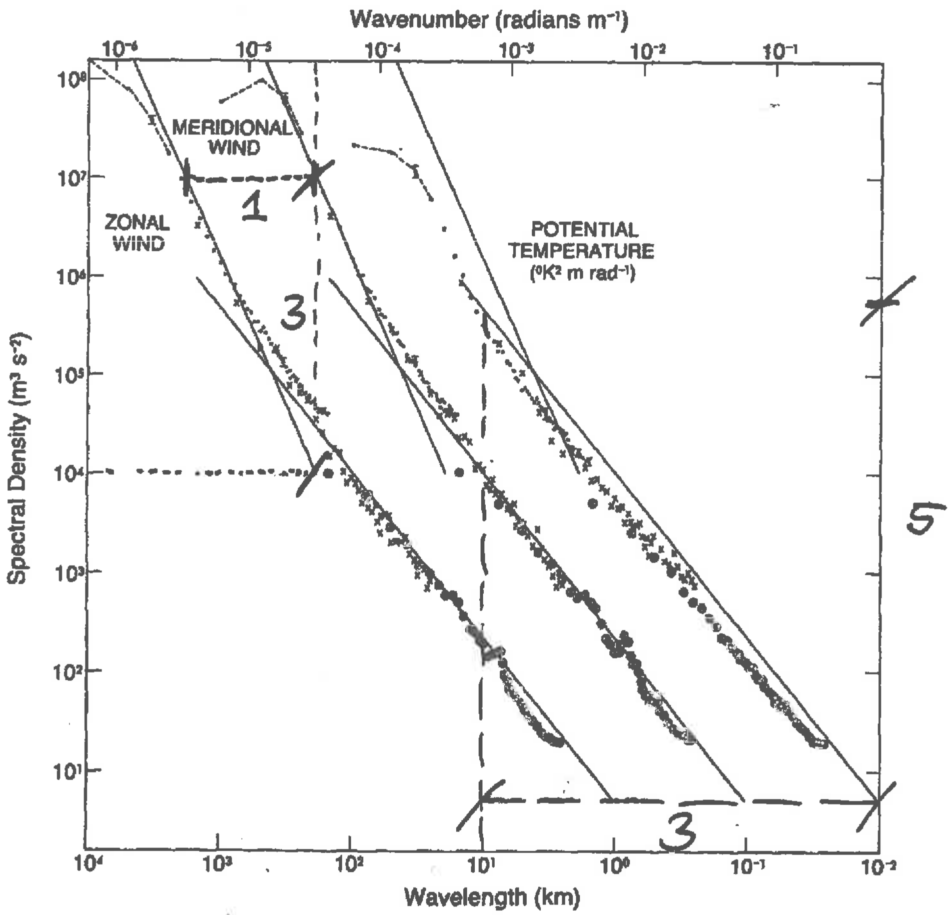


FIG. 3. Variance power spectra of wind and potential temperature near the tropopause from GASP aircraft data. The spectra for meridional wind and temperature are shifted one and two decades to the right, respectively; lines with slopes and are entered at the same relative coordinates for each variable for comparison.

[1]From G.D. Nastrom and K.S. Gage, 1985: *A Climatology of Atmospheric Wavenumber Spectra of Wind and Temperature Observed by Commercial Aircraft.* J. Atmos. Sci. 42, 950–960. (Redacted.)

They claim that there are two regimes, in each of which there is an allometric (power-law) relation between E and k, corresponding to straight lines on the log-log plot. I.e., in each of the regimes,

$$E(k) = c \cdot k^{\alpha}$$

$$\Rightarrow \log_{10} E = \log_{10} c + \alpha \log_{10} k$$

We see that the allometric exponent is given by the slope of the lines.

<u>Big structures</u> (wavelength $\gtrsim 0.5 \cdot 10^3$ km $= 500$ km): slope $= -3$

$$\Rightarrow E(k) = c_1 \cdot k^{-3}$$

<u>Small structures</u> (wavelength $\lesssim 500$ km): slope $= -\frac{5}{3}$

$$\Rightarrow E(k) = c_2 \cdot k^{-\frac{5}{3}}$$

Note that the graph shows three sets of measurements which behave in exactly the same way. For clarity, 2 of them have been shifted to the right, cf. figure caption.

2. Consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

(a) For which values of x is the function continuous, and why?

(b) Can you redefine f at the point(s) of discontinuity to make it continuous everywhere?

(c) Compute $f'(x)$ whereever it is defined.

(5+5+5)

(a) At $x = 1$, the function is not defined. Otherwise, it is continuous as a composition of continuous functions.

(b) $$f(x) = \frac{(x+1)(x-1)}{(x-1)} = x+1 \qquad (*)$$

$$\Rightarrow \lim_{x \to 1} f(x) = 2$$

So, setting $f(1) := 2$ yields a function which is continuous everywhere.

(c) Using $(*)$, $f'(x) = 1$ (for $x \neq 1$).

3. Find the equation of the tangen line at $x = e$ for the graph of the function

$$f(x) = \ln(\ln x)\,.$$

(10)

$$f(e) = \ln(\ln e) = \ln 1 = 0$$

$$f'(x) = \frac{1}{x}\,\frac{1}{\ln x} \quad \Rightarrow \quad f'(e) = \frac{1}{e}\,\frac{1}{\ln e} = \frac{1}{e}$$

$$\Rightarrow \quad \ell(x) = e^{-1}(x-e) = \frac{x}{e} - 1$$

is the tangent line equation.

4. Consider the function

$$f(x) = \frac{1}{1+e^x}$$

For which values of x is the function defined? Find the vertical and horizontal asymptotes (if any), find and classify all critical points, determine where the function is concave up or concave down, find all points of inflection, and sketch the graph into the coordinate system provided. (5+5+5+5+5)

- $e^x > 0$, so f is defined everywhere.

  $\Rightarrow$ no vertical asymptotes.

- $\lim_{x\to\infty} \frac{1}{1+e^x} = 0$ $\quad\Rightarrow y=0$ is hor. asympt. as $x \to \infty$

  $\lim_{x\to -\infty} \frac{1}{e^x+1} = \frac{1}{1} = 1$ $\quad\Rightarrow y=1$ is hor. asympt. as $x \to -\infty$

- $f'(x) = -\frac{e^x}{(1+e^x)^2} < 0$ $\quad\Rightarrow$ no critical points, f decreasing everywhere.

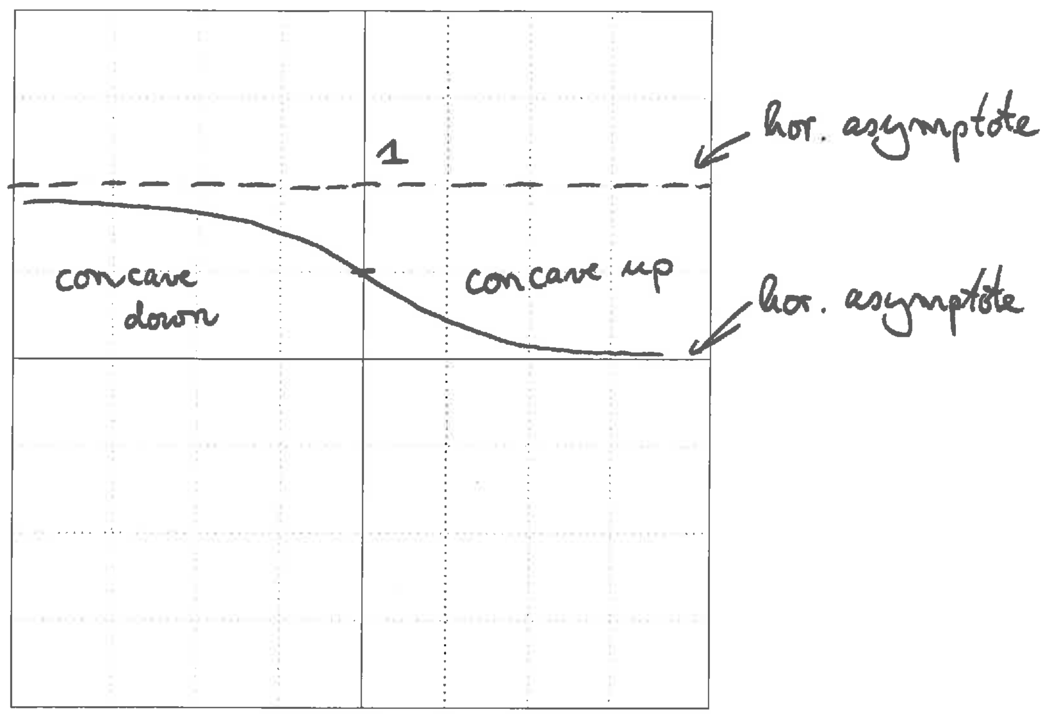
- $f''(x) = -\frac{e^x(1+e^x)^2 - 2e^x(1+e^x)e^x}{(1+e^x)^4} = \frac{e^x}{(1+e^x)^3}\underbrace{(2e^x-(1+e^x))}_{=e^x-1}$

So $f''(x)=0 \Leftrightarrow e^x = 1 \Leftrightarrow x=0$ (pt. of inflection)

$f''(x)<0$ for $x<0$ $\Rightarrow f$ concave down

$f''(x)>0$ for $x>0$ $\Rightarrow f$ concave up.

$\Rightarrow$ there is a point of inflection at $(0, f(0)) = (0, \frac{1}{2})$.



1
hor. asymptote
concave down
concave up
hor. asymptote

5. A company has determined that the price per unit p for one of its products and the demand x are related by

$$p(x) = 1\,500 - x$$

The cost for producing x units of the product is

$$C(x) = 200\,000 + 500\,x$$

(a) Write out the company's profit as a function of demand x.

(b) What price per unit should the product be sold at to maximize profit?

(5+10)

(a) Revenue $R(x) = x\,p(x) = 1500x - x^2$

Profit $P(x) = R(x) - C(x)$

$= 1500x - x^2 - 200\,000 - 500x$

$= -x^2 + 1000x - 200\,000$

(b) $P'(x) = -2x + 1000$

$\Rightarrow$ critical point when $-2x + 1000 = 0$ $\Rightarrow x = 500$

This corresponds to global maximum as P is a parabola open downward. Then

$$p(x) = 1000$$

This is the optimal price per unit.

6. (a) If you measure $x = 3 \pm 1$ and $y = 4 \pm 1$ with independent uncertainties, what should you report for

$$z = \frac{xy}{x+y}$$

together with its uncertainty?

(b) In the setting of (a), show that the uncertainties of x, y, and z satisfy the relation

$$\frac{\Delta z^2}{z^4} = \frac{\Delta x^2}{x^4} + \frac{\Delta y^2}{y^4}.$$

(10+5)

(a) $z = \frac{3 \cdot 4}{3+4} = \frac{12}{7}$

$\frac{\partial z}{\partial x} = \frac{y(x+y) - 1 \cdot xy}{(x+y)^2} = \frac{y^2}{(x+y)^2}$ here: $\frac{\partial z}{\partial x} = \frac{16}{49}$

$\frac{\partial z}{\partial y} = \frac{x^2}{(x+y)^2}$ by symmetry. $\frac{\partial z}{\partial y} = \frac{9}{49}$

$\Rightarrow (\Delta z)^2 \approx \left(\frac{\partial z}{\partial x} \Delta x\right)^2 + \left(\frac{\partial z}{\partial y} \Delta y\right)^2 = \left(\frac{16}{49}\right)^2 + \left(\frac{9}{49}\right)^2 = \frac{337}{49^2}$

$\Rightarrow \Delta z \approx \frac{\sqrt{337}}{49}$ $\Rightarrow z = \frac{12}{7} \pm \sqrt{\frac{337}{49}}$

(b) $\frac{\Delta z^2}{z^4} \approx \frac{(x+y)^4}{x^4 y^4} \left[\frac{y^4}{(x+y)^4} \Delta x^2 + \frac{x^4}{(x+y)^4} \Delta y^2 \right]$

$= \frac{\Delta x^2}{x^4} + \frac{\Delta y^2}{y^4}$

7. Consider

$$f(x, y) = x^2 + 2x + 5 - 4y + y^2 .$$

(a) Find the global minimum of f.

(b) Justify explicitly why the answer you obtained in part (a) is indeed the global minimum.

(10+5)

(a) $\frac{\partial f}{\partial x} = 2x + 2 = 0 \Rightarrow x = -1$

$\frac{\partial f}{\partial y} = -4 + 2y = 0 \Rightarrow y = 2$

The minimum is at $(-1, 2)$ where $f(x,y) = (-1)^2 + 2(-1) + 5 - 4\cdot 2 + 2^2$

$= 1 - 2 + 5 - 8 + 4 = 0$

(b) As $f(x,y) \to \infty$ as either $x \to \infty$ or $y \to \infty$, the only critical point must correspond to a global minimum.

Alternatively, observe that $f(x,y) = (x+1)^2 + (y-2)^2 \geq 0$,

so the only root of f must coincide with the global minimum.

8. Compute the following integrals:

(a) $\int_{-1}^{1} x^{67}\,dx$

(b) $\int_{1}^{2} \frac{x}{\sqrt{x^2+1}}\,dx$

(c) $\int x \sin x\,dx$

(5+5+5)

(a) $\int_{-1}^{1} x^{67}\,dx = 0$ by oddness of the integrand.

Alternatively: $\int_{-1}^{1} x^{67}\,dx = \frac{1}{68} x^{68} \Big|_{-1}^{1} = \frac{1}{68}\left(1^{68} - (-1)^{68}\right) = 0$

(b) $\int_{1}^{2} \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int_{2}^{5} \frac{du}{\sqrt{u}}$ $u = x^2+1$ $\Rightarrow du = 2x\,dx$

$= \frac{1}{2} u^{\frac{1}{2}} \Big|_{2}^{5} = \sqrt{5} - \sqrt{2}$

(c) $\int x \sin x\,dx = x(-\cos x) - \int 1 \cdot (-\cos x)\,dx$

$= -x\cos x + \sin x + C$

9. (a) Solve the differential equation

$$\frac{dy}{dt} = 1 - y$$

with $y(0) = 0$.

(b) What is $\lim_{t \to \infty} y(t)$ for your solution from part (a)?

Note: it is possible to solve (b) independent of (a) and use the result to double-check the computation in (a).

(10+5)

(a) $\frac{dy}{1-y} = dt \quad \Rightarrow \quad \int_0^{y(t)} \frac{dy}{1-y} = \int_0^t dt$

$\Rightarrow \quad -\ln(1-y) \Big|_0^{y(t)} = t$

$\Rightarrow \quad \ln(1-y(t)) = -t$

$\Rightarrow \quad 1-y(t) = e^{-t}$

$\Rightarrow \quad y(t) = 1 - e^{-t} \qquad (*)$

(b) $\lim_{t \to \infty} y(t) = 1$ (directly from $(*)$)

Note: $\frac{dy}{dt} = 0$ whenever $y = 1$, so this is a fixed point. It is stable because $\frac{dy}{dt} < 0$ when $y > 1$ and $\frac{dy}{dt} > 0$ when $y < 1$. This implies $\lim_{t \to \infty} y(t) = 1$.