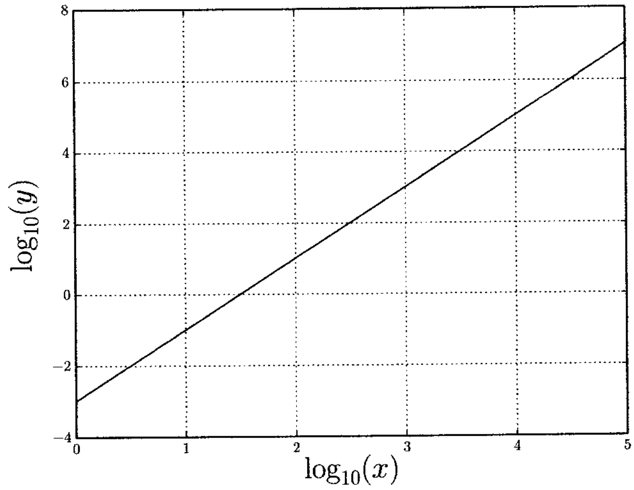
1. For each graph, determine y as a function of x.

(a)



$\log_{10} y = m \log_{10} x + b$

Take points $(0,-3)$ and $(1,-1)$ on the line:

$-3 = m \cdot 0 + b \Rightarrow b = -3$

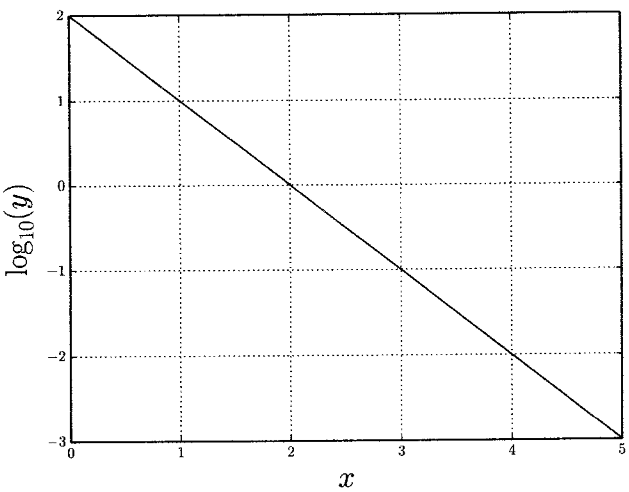
$-1 = m \cdot 1 + b \Rightarrow m = 2$

$\Rightarrow \log_{10} y = 2 \log_{10} x - 3$

$\Rightarrow y = 10^{-3} x^2$

(Lines on a log-log plot always indicate an allometric relation!)

(b)



$\log_{10} y = mx + b$

Take points $(0,2)$ and $(1,1)$:

$2 = m \cdot 0 + b \Rightarrow b = 2$

$1 = m + b \Rightarrow m = -1$

$\Rightarrow \log_{10} y = -x + 2$

$\Rightarrow y = 10^{2-x} = 100 \cdot 10^{-x}$

~~(10+10)~~
(5+5)

(Lines on a semilog plot always indicate an exponential relation!)

2. Find first and the second derivative of the function

$$f(x) = \ln(\sin x)\,.$$

Simplify your result by recalling from class that $\sin^2 x + \cos^2 x = 1$. (5+5+5)

$$f'(x) = \frac{\cos x}{\sin x}$$

$$f''(x) = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

3. Find the minimum and the maximum value of the function

$$f(x) = x^3 - 3x^2 + 3x - 1$$

on the interval $0 \leq x \leq 2$. (10)

Critical points:

$f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$

Thus, f has a critical point at $x=1$, but $f'(x) \geq 0$ for all values of x. I.e., it does not change sign. So no extreme value at the critical point.

Testing the end points of the interval $[0,2]$:

$f(0) = -1$

$f(2) = 8 - 12 + 6 - 1 = 1$

$\Rightarrow$ The minimum of f on $[0,1]$ is -1, the maximum is 1.

4. Consider the function

$$f(x) = \ln(4 - x^2)$$

For which values of x is the function defined? Find the vertical and horizontal asymptotes (if any), find and classify all critical points, determine where the function is concave up or concave down, find all points of inflection, and sketch the graph into the coordinate system provided. (5+5+5+5+5)

- The logarithm is only defined when $4 - x^2 > 0$

  $\Rightarrow 4 > x^2 \quad \Rightarrow -2 < x < 2$.

- No horizontal asymptotes because f is only defined on a bounded interval.

- $\lim\limits_{\substack{x \to 2 \\ x < 2}} f(x) = -\infty$; $\quad \lim\limits_{\substack{x \to -2 \\ x > -2}} f(x) = -\infty$

  $\Rightarrow$ vertical asymptotes at $x = \pm 2$.

- $f'(x) = \dfrac{-2x}{4 - x^2}$ $\Rightarrow$ critical point at $x = 0$ where f' changes from +ve to −ve.

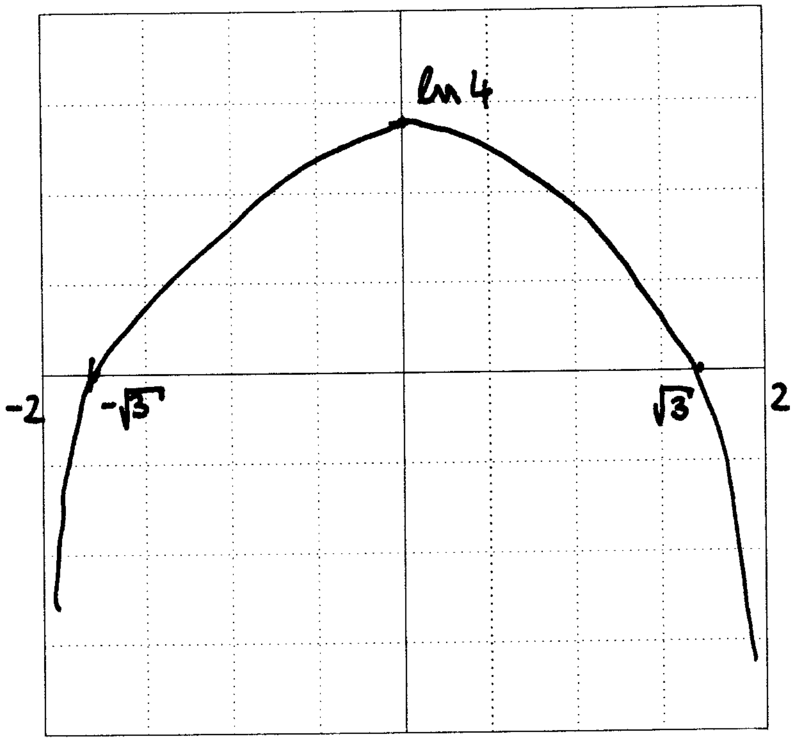
  $\Rightarrow$ maximum at $x = 0$, $y = \ln 4$

- $f''(x) = \dfrac{-2(4-x^2) - (-2x)(-2x)}{(4-x^2)^2} = -\dfrac{8 + 2x^2}{(4-x^2)^2} < 0$

  $\Rightarrow$ no points of inflection, f is always concave down.

- For sketching (not strictly necessary): $f(x) = 0 \Rightarrow 4 - x^2 = 1$

  $\Rightarrow x^2 = 3 \quad \Rightarrow x = \pm\sqrt{3}$


ln 4
-2
-√3
√3
2

5. The marketing department has determined that demand D (defined as the number of units sold) for a new product is decreasing exponentially with price p, i.e.,

$$D(p) = N e^{-rp}$$

for some constants N and r. How much should you charge to maximize revenue? Does your solution formula depend on N? State a reason for this in economic terms. (15)

For the revenue, we have

$$R(p) = p\, D(p) = N p\, e^{-rp}$$

To find the maximum, we compute

$$R'(p) = N e^{-rp} - rNp\, e^{-rp} = (1-rp)\, N e^{-rp},$$

we have a critical point when $1 - rp = 0 \Rightarrow \boxed{p = \frac{1}{r}}$.

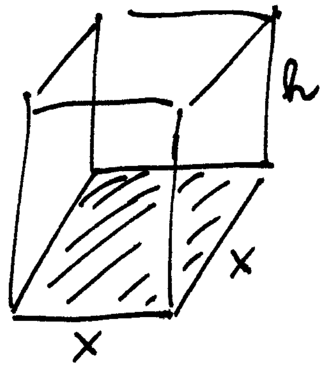
It corresponds to a maximum since $R(0) = 0$, $R(p) > 0$ for $p > 0$ and $R(p) \to 0$ as $p \to \infty$.

The optimal price does not depend on N. This can be understood as follows. Since $D(0) = N$, we can think of N as the number of customers for which the product has any value, or the size of the market. Thus, we can write

$$R(p) = N \cdot (p\, e^{-rp}) = \text{market size} \cdot \text{expected revenue per customer}.$$

Since N is given, optimizing R corresponds to optimizing the second factor!

6. A box with a square base and open top must be constructed using no more than $2\,\mathrm{m}^2$ of material. Find the dimensions of the box that maximize its volume. (15)



$$V = h x^2$$

$$A = \underbrace{4xh}_{\text{sides}} + \underbrace{x^2}_{\text{bottom}}$$

$$\Rightarrow \quad h = \frac{A - x^2}{4x}$$

$$\Rightarrow V(x) = \frac{1}{4}(A - x^2)x = \frac{1}{4}(Ax - x^3)$$

$$V'(x) = \frac{1}{4}(A - 3x^2)$$

$\Rightarrow$ critical point at $A = 3x^2$ or $x = \sqrt{\frac{A}{3}}$

(neg. root does not make sense)

Since $V''(x) = -\frac{3}{2}x < 0$ for $x > 0$, this corresponds to a maximum.

Correspondingly, $h = \dfrac{A - \frac{A}{3}}{4\frac{\sqrt{A}}{\sqrt{3}}} = \dfrac{\sqrt{3A}}{6} = \dfrac{1}{2}\sqrt{\dfrac{A}{3}}$.

For $A = 2\,\mathrm{m}^2$, we get $x = \sqrt{\frac{2}{3}}$ m, $h = \frac{1}{2}\sqrt{\frac{2}{3}}$ m.