Gaussian elimination revisited

solve $\quad Ax = b \qquad A \in \mathrm{Mat}(n \times m)$

$\qquad\qquad\qquad\qquad b \in \mathbb{R}^n$

here, in particular, A a <u>wide</u> matrix, i.e. $m > n$

$\Rightarrow$ system is underdetermined, many solutions

In Finite Mathematics: least-norm solution, i.e. a solution that minimizes $\|x\|$.

In this class: goal is to find a solution that optimizes a <u>linear</u> objective function.

To start, revisit the solution structure/procedure, by example:

$$A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 2 & 6 & 0 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 2 & 6 & 0 & -1 & 1 \end{array}\right) \xrightarrow{-2R1+R2 \to R2} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 0 & 0 & -2 & -3 & -3 \end{array}\right)$$

$$\begin{array}{ccccc} & & 1 & \frac{3}{2} & \frac{3}{2} \end{array}$$

columns with pivots

$$\xrightarrow{R1 - R2 \to R1} \left(\begin{array}{cccc|c} 1 & 3 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & (-1)\,0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & (-1)\,0 & 0 \end{array}\right) \longrightarrow \ldots$$

$$x = \underbrace{\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{3}{2} \\ 0 \end{pmatrix}}_{\text{particular solution}} + \underbrace{\lambda \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{3}{2} \\ -1 \end{pmatrix}}_{\text{Two vectors spanning the space of solutions to } Ax=0 \text{ (homogeneous solutions)}}$$

What is a maximal set of l.i. columns of A?

E.g. $B = \{1, 3\}$ are indices of l.i. colums (eliminated matrix has pivots here)

The particular solution $\begin{pmatrix} \frac{1}{2} \\ 0 \\ \vdots \end{pmatrix}$ is called a basic solution, having $x_i = 0$ if $i \notin B$

components not indexed in B

Q: Are there other basic solutions, e.g. corresponding to $B = \{2,3\}$?

$$\cdots \to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \cdots$$

get another parametric representation of the solution set:

$$x = \underbrace{\begin{pmatrix} 0 \\ \frac{1}{6} \\ \frac{3}{2} \\ 0 \end{pmatrix}}_{\text{particular solution, basic!}} + \underbrace{\tilde{\lambda} \begin{pmatrix} -1 \\ \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -\frac{1}{6} \\ \frac{3}{2} \\ -1 \end{pmatrix}}_{\text{these vectors span the space of solutions to hom. eqn.}}$$

Another option: $B = \{2,4\}$

$$\cdots \to \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 1 & 1 \end{array}\right) \xrightarrow{\frac{1}{6}R4 + R2 \to R2} \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & \frac{1}{9} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 1 & 1 \end{array}\right)$$

$$x = \begin{pmatrix} 0 \\ \frac{1}{3} \\ 0 \\ 1 \end{pmatrix} + \tilde{\tilde{\lambda}} \begin{pmatrix} -1 \\ \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + \tilde{\tilde{\mu}} \begin{pmatrix} 0 \\ \frac{1}{9} \\ -1 \\ \frac{2}{3} \end{pmatrix}$$

particular, basic solution

Now back to LP:

Fact: Every linear program can be written in the standard form

minimize $c^T x$ with $c \in \mathbb{R}^m$

subject to $Ax = b$ $\quad A \in \text{Mat}(n \times m)$

$b \in \mathbb{R}^n$

$x \geq 0$ $\qquad\qquad$ $b \in \mathbb{R}$

meaning $x_j \geq 0$ for $j = 1, \dots, n$

"<u>Proof by example</u>":

maximize $\quad z = x_1 + 2x_2 + 3x_3$ $\qquad$ ① → minimize $\quad -x_1 - 2x_2 - 3x_3$

subject to
$$x_1 + x_2 - \overset{u-v}{\overset{\shortparallel}{x_3}} = 1$$

③ → $x_1 + x_2 - u + v = 1$

$$-2x_1 + x_2 + 2x_3 \geq -5$$

① → $2x_1 - x_2 - 2x_3 \leq 5$

$$x_1 - x_2 \leq 4$$
$$x_2 + x_3 \leq 5$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

→ ⊛

<u>Steps:</u> ① Turn maximization into minimization and write inequalities in standard order.

② Introduce "slack variables" to turn inequalities into equalities + nonneg. constraint:

⊛ can be written

$$2x_1 - x_2 - 2x_3 + s_1 = 5, \quad s_1 \geq 0$$
$$x_1 - x_2 + s_2 = 4, \quad s_2 \geq 0$$
$$x_2 + x_3 + s_3 = 5, \quad s_3 \geq 0$$

③ Replace variables without non-negativity constraint by differences:

$$x_3 = u - v \qquad \text{where } u \geq 0, \; v \geq 0$$

Then:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ u \\ v \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} \qquad A = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & u & v & s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 2 & -1 & -2 & 2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad b = \begin{pmatrix} 1 \\ 5 \\ 4 \\ 5 \end{pmatrix}$$

$$c = \begin{array}{cc} \begin{pmatrix} -1 \\ -2 \\ -3 \\ +3 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{array}{c} x_1 \\ x_2 \\ u \\ v \\ s_1 \\ s_2 \\ s_3 \end{array} \end{array}$$