Recall: ① Any particular solution to $Ax = b$ obtained via the standard process of Gaussian elimination is a basic solution: If B is the set of column indices corresponding of a basis of the column space (i.e. columns with pivots), then a basic solution has $x_j = 0$ if $j \notin B$.

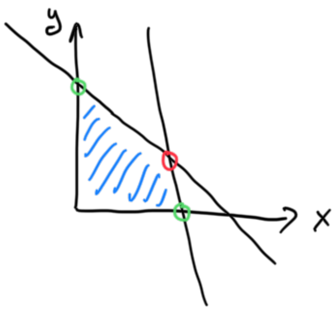
Note: B can usually be chosen in many different ways

② Every LP problem can be written in the standard form

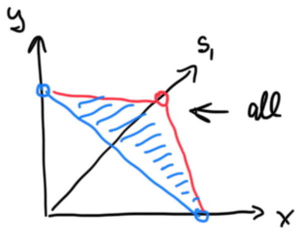
$$\min \quad z = c^T x$$
$$\text{subject to} \quad Ax = b$$
$$x \geq 0$$

Q: how does the feasible region look like?

Recall graph. solution example:



feasible region is a polygon



← all corners are basic feasible solutions

Theorem: If a standard-form LP has optimal solutions, then there is an optimal basic solution. (I.e. an optimal solution that is a vertex of the feasible region.)

Proof: Suppose x is optimal, but not basic.

We can assume that all components of x are non-zero (otherwise just delete them – that does not change the problem)

$\Rightarrow$ There must be at least one direction vector $v \neq 0$, so that $x \pm v$ is feasible.

$$A(x+v) = b \quad \Rightarrow \quad \underbrace{Ax + Av = b}_{=} \quad \Rightarrow \quad Av = 0$$

Since x is optimal,

$$\left.\begin{array}{l} c^T x \leq c^T(x+v) = c^T x + c^T v \quad \Rightarrow \quad c^T v \geq 0 \\ c^T x \leq c^T(x-v) = c^T x - c^T v \quad \Rightarrow \quad c^T v \leq 0 \end{array}\right\} \quad c^T v = 0$$

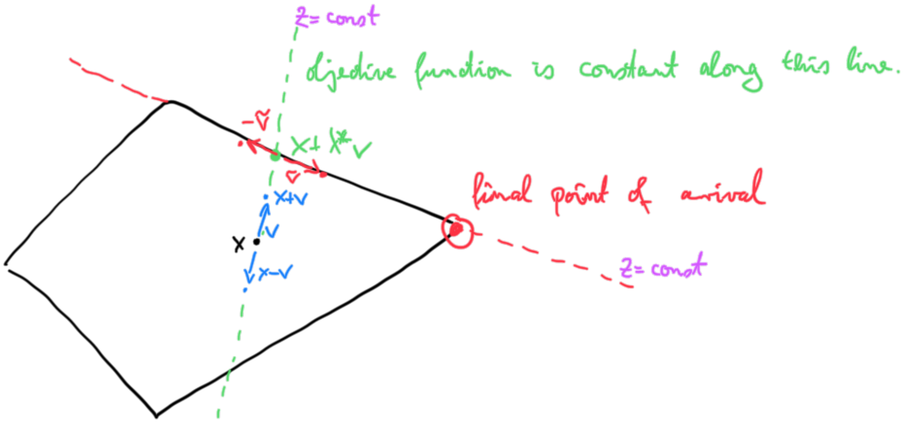
v has at least one negative component (if not, just replace v by $-v$)

$x + \lambda v$:
- for $\lambda \in [0,1]$, $x + \lambda v$ is feasible
- $c^T(x+\lambda v) = c^T x + \lambda \underbrace{c^T v}_{=0} = c^T x \quad \Rightarrow \quad x + \lambda v$ is optimal

Let's increase λ: at some value $\lambda = \lambda^*$, one component of $x + \lambda v$ will change sign from + to −, thus leaving the feasible region.

$x + \lambda^* v$ is still feasible, optimal, has at least one component that is 0.

Now iterate process until solution is <u>basic</u>.



Recall:
$$\left.\begin{array}{l} Ax = b \\ A(x+v) = b \end{array}\right\} \quad Av = 0$$

<u>Conclusion:</u> To find optimal solutions, it suffices to check all basic feasible solutions.

Good news: Gaussian elimination naturally gives basic solutions

Bad news: In general, there are <u>many</u> basic feasible solutions, often too many to list and check.

<u>Solution</u>: "Simplex Algorithm"

(i) Start with any basic feasible solution

(ii) Swap one basic variable ("leaving variable") for another variable ("entering variable") s.t. objective function improves the most.

(iii) If this cannot be done, stop; otherwise repeat.

<u>Note</u>: $z = c^T x$ (objective function)

can be written $\underline{c^T x} - z = \underline{0}$ $\qquad\qquad Ax = b$

So this leads to writing out a "simplex tableau"

$$
\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
\hline
 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
A & 2 & -1 & -2 & \boxed{2} & 1 & 0 & 0 & 5 \\
 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 4 \\
 & 0 & 1 & 1 & \boxed{-1} & 0 & 0 & 1 & 5 \\
\hline
c^T & -1 & -2 & -3 & \boxed{3} & 0 & 0 & 0 & \boxed{0}
\end{array}
\quad b
$$

after elimination, this tracks $-z$

not zero, need to be eliminated.

$$
\begin{array}{l|ccccccc|c}
 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
-2R1+R2 \to R2 & 0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\
 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 4 \\
R1+R4 \to R4 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 6 \\
\hline
-3R1+R5 \to R5 & -4 & -5 & 0 & 0 & 0 & 0 & 0 & -3
\end{array}
$$

This is representing a basic feasible solution with

$x_1 = 0,\ x_2 = 0,\ u = 0,\ v = 1,\ s_1 = 3,\ s_2 = 4$

$s_3 = 6,\quad z = 3$