Towards the dual LP problem

"Car factory problem" (like WYNDOR):

- produce 2 products: cars (x_1) and trucks (x_2)

maximize $z = 3x_1 + 2x_2$

(← profit per truck; ↑ profit per car)

subject to

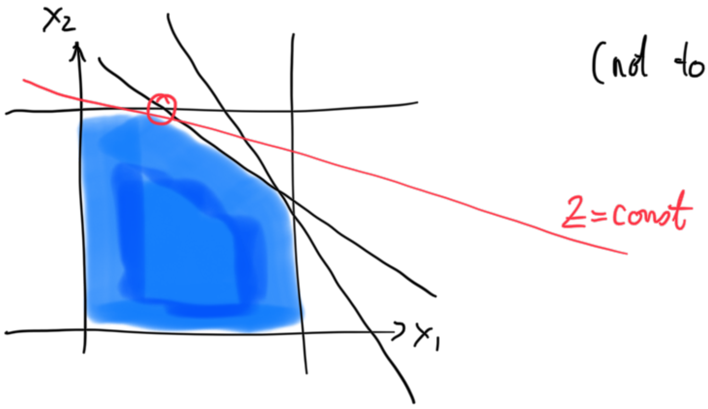
$5x_1 \leq 100$ "car assembly" (← 100h available; ↑ one car needs 5h assembly)

$10x_2 \leq 100$ "truck assembly"

$4x_1 + 3x_2 \leq 100$ "metal stamping process"

$3x_1 + 5x_2 \leq 100$ "engine assembly"

$x \geq 0$



(not to scale)

Via simplex method:

x_1	x_2	s_1	s_2	s_3	s_4	
5 (1)		1 ($\frac{1}{5}$)				100 (20)
	10		1			100
4 (0)	3 (3)	$-\frac{4}{5}$		1 (1)		100 (20)
3 (0)	5 (5)	$-\frac{3}{5}$			1 (1)	100 (40)

$5x_1 \leq 100$

$\Leftrightarrow 5x_1 + s_1 = 100$

-3_0	-2_{-2}	$\frac{3}{5}$					0_{60}

$R4 + 3R1 \to R4$

Now x_2 is new entering variable, new pivot is in third row

x_1	x_2	s_1	s_2	s_3	s_4	
1		$\frac{1}{5}$				20
0	0	*	1	*	0	$100 - \frac{200}{3}$
	1	$-\frac{4}{15}$		$\frac{1}{3}$		$\frac{20}{3}$
	0	*		*	1	$40 - 5\,\frac{20}{3}$
0	0	>0	0	>0	0	$60 + 2 \cdot \frac{20}{3}$

$= 73,\dots$

Look at structure:

$n = 2$ physical variables
$m = 4$ constraints
} $n + m = 6$ variables altogether

Unless there is some redundancy, there will be m basic variables.

Typically, all physical variables are basic (assume for simplicity),

so we have $m - n$ slack variables that are basic.

Q: How many slack variables are non-basic? n !!

The non-basic slack variable point to the binding constraints

↑ satisfied with equality

Typically: as many binding constraints as physical variables.

Crossing out non-binding constraints and non-basic slacks does not change the solution, so it can be written

$$\tilde{A} x = \tilde{b} \qquad \text{where } \tilde{A} = \begin{pmatrix} 5 & 0 \\ 4 & 3 \end{pmatrix}$$

$\sim$ (100)

$$b = \begin{pmatrix} 100 \end{pmatrix}$$

are the remaining coefficients

$$\Rightarrow \quad x = \tilde{A}^{-1}\tilde{b}$$

$$z = c^T x = \underbrace{c^T \tilde{A}^{-1}}_{\tilde{y}^T} \tilde{b} \qquad \tilde{y} \in \mathbb{R}^2$$

$$= y^T b \qquad y \in \mathbb{R}^4 \quad \text{fill up } \tilde{y} \text{ with zeros}$$

Recall abstract form of problem:

$$\max \quad c^T x$$

$$\text{subject to} \quad Ax \le b$$

$$x \ge 0$$

Q: How does the profit change if I change the capacities b by a small amount?

binding vs. non-binding constraints remain the same.

Replace b by $b + \delta$:

New solution: $x = \tilde{A}^{-1}(\tilde{b} + \tilde{\delta})$

$$z(\delta) = y^T (b + \delta) = \underbrace{y^T b}_{z(0)} + y^T \delta$$

$$\Rightarrow z(\delta) = z(0) + \underbrace{y^T \delta}_{y_1 \delta_1 + y_2 \delta_2 + \dots}$$

↑ shadow prices

The shadow price of resource i is the change of profit per unit of capacity

under current operating conditions

at current operating conditions.

Theorem: The value of a company (in term of profits from its operation) equals the value of all its resources valued at current shadow prices.

Proof: Consider $\delta = -b$ (Note: This change seems "big" but it can be done via proportional rescaling of all resources which does not change the shape, only the size, of the feasible region.)

$$z(\delta) = 0 \qquad \text{(no resources, no operation, so no profit)}$$

$$\Rightarrow \quad 0 = z(\delta) = z(0) + y^T \underbrace{\delta}_{=-b}$$

$$\Rightarrow \quad \boxed{z(0)} = \boxed{y^T b}$$

Scenario: Company X wants to buy a bundle of resources.

Q: how to fairly price the resources?

In the example, we produce cars and trucks.

For cars: $5y_1 + 4y_3 + 3y_4 \geq 3$

For trucks: $10y_2 + 3y_3 + 5y_3 \geq 2$

minimize y^T

This leads to the LP, the dual problem

$$\begin{aligned} &\text{minimize} && y^T b \\ &\text{subject} && A^T y \geq c \\ & && y \geq 0 \end{aligned}$$