Calculus and Elements of Linear Algebra I

Homework 3

Due on Moodle, September 28, 2020

1. Compute the following limits:

   (a) $\lim\limits_{\theta\to 0} \dfrac{\theta}{\sin\theta}$

   (b) $\lim\limits_{\phi\to 0} \dfrac{1-2\,\cos^2\phi}{\phi}$

   Hint: For (b), use the double-angle identity

   $$\cos^2\phi = \frac{1+\cos 2\phi}{2}\,.$$

2. The (natural) exponential function can be defined as

   $$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\,. \qquad (*)$$

   Use this definition to show that $\exp(a+b) = \exp(a)\cdot\exp(b)$ for all $a, b \in \mathbb{R}$.

   Hint: Convert the definition (*) into a different limit by setting $y = x/n$.

3. Show that the equation

   $$x = \cos x$$

   has at least one solution in the interval $(0, \pi/2)$.

   Hint: Intermediate value theorem.

4. Let $f\colon \mathbb{R}\to\mathbb{R}$ be continuous and $g\colon \mathbb{R}\to\mathbb{R}$ some real-valued function. Show, by giving a counter example, that the following statement is not generally true: *If $h(x) = f(g(x))$ is continuous, then g is also continuous.*

5. Compute the derivative of the following functions directly from the definition

   $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\,.$$

   (a) $f(x) = x^2$

   (b) $f(x) = \sqrt{x}$