Calculus and Elements of Linear Algebra I

Homework 8

Due on Moodle, Monday, November 9, 2020

1. Find
$$\int_0^1 \ln x \,\mathrm{d}x\,.$$

2. Show that
$$\int_0^\infty \frac{1}{\sqrt{1+x^8}}\,\mathrm{d}x$$
is convergent.

   Hint: There is no elementary way to evaluate this integral. However, to only *test* convergence, you can bound the integrand by a simpler function and use the following fact without proof: Let $f\colon [a,\infty) \to \mathbb{R}$ be a bounded and increasing function. Then $\lim_{x\to\infty} f(x)$ exists.

3. (a) Show that the volume enclosed when revolving the curve $y = f(x)$, where $f\colon [a,b] \to [0,\infty)$ about the x-axis in three-dimensional x-y-z space is given by
$$V = \pi \int_a^b f^2(x)\,\mathrm{d}x\,.$$
   (b) Compute the volume of the solid obtained by revolving the graph of of $y = x^3$ on $[0,1]$ about the x-axis.

4. (a) Compute the volume of the infinite solid obtained by revolving the graph of $y = x^{-2/3}$ on $[1,\infty)$ about the x-axis and show that it is finite.
   (b) Show that the cross-sectional area when cutting this solid in the x-y plane is infinite.

5. Hook's law states that the force exerted by an ideal spring when extended from its equilibrium position at $x = 0$ to length x is given by
$$F(x) = -k\,x\,,$$
where k is a positive constant characterizing the stiffness of the spring. Compute the work required to expand the spring from its equilibrium position to length ℓ.