Numbers

$\mathbb{N} = \{1, 2, 3, \dots\}$ "natural numbers" $\rightarrow$ Peano axioms

Key operations: $+$, $\cdot$

commutative: $a + b = b + a$ $\quad \forall a, b \in \mathbb{N}$

$a \cdot b = b \cdot a$ "for all a and b in the natural numbers"

associative: $a + (b+c) = (a+b) + c \quad \forall a, b, c \in \mathbb{N}$

$a(bc) = (ab)c$

distributive: $a(b+c) = ab + ac \quad \forall a, b, c \in \mathbb{N}$

Goal: extend number set to meet certain criteria while keeping these laws true.

Neutral elements: $1 \cdot a = a \quad \forall a \in \mathbb{N}$ "1 is the neutral element of multiplication"

no neutral element for "+" yet. So define 0 as any object $\notin \mathbb{N}$ s.t.

$a + 0 = a \quad \forall a \in \mathbb{N}$

Also want to keep the distributive law:

$$\underbrace{b\,\underbrace{(a+0)}_{a}}_{ba} = ba + \underline{\underline{b0}} \qquad \Longrightarrow b0 = 0$$

$\longrightarrow \mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0, 1, 2, \dots\}$

union between sets

Solving equations: $a + x = b$ for given $a, b \in \mathbb{N}_0$ $\quad (*)$

cannot be solved if $a > b$

$\rightarrow$ introduce negative numbers: $a \in \mathbb{N}$: $-a \notin \mathbb{N}$ is an object that satisfies $-a + a = 0$

Now we can solve $(*)$: $x = b - a$

$\mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}$ "integers" [where $-0 = 0$]

$\longrightarrow \mathbb{Z} =$

For multiplication: $\quad ax = b \qquad a, b \in \mathbb{Z}$

cannot be solved in $\mathbb{Z}$ in general: E.g. $3x = 2$

$\longrightarrow$ introduce rational numbers via multiplicative inverse:

$a \in \mathbb{Z} \setminus \{0\}$ we <u>define</u> $\frac{1}{a}$ as the object such that $a \cdot \frac{1}{a} = 1$

and write $\frac{a}{b} = a \cdot \frac{1}{b}$

$\longrightarrow \mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z},\ b \in \mathbb{Z} \setminus \{0\} \right\}$ "rational numbers"

<u>Powers (here: integer powers)</u>

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n\text{-times}}$$

polynomials: $\quad p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

where $a_0, \ldots, a_n \in \mathbb{Q}$ (for now)

are given numbers and x is an argument

<u>E.g.</u>: $3x^5 + 2x - 10 = p(x)$ is a polynomial

<u>Roots</u>: values for x where $p(x) = 0$.

E.g. if $n = 1$: (n is called "degree"): $\quad a_1 x + a_0 = 0 \Rightarrow x = -\frac{a_0}{a_1}$

$n = 2$: $\quad a_2 x^2 + a_1 x + a_0 = 0$ "quadratic equation"

$$\Rightarrow 4a_2^2 x^2 + 4a_1 a_2 x + 4a_0 a_2 = 0$$

$$\Rightarrow (2a_2 x + a_1)^2 - a_1^2 + 4a_0 a_2 = 0$$

$$\Rightarrow (2a_2 x + a_1)^2 = a_1^2 - 4a_0 a_2$$

$$\Rightarrow 2a_2 x + a_1 = \pm \sqrt{a_1^2 - 4a_0 a_2}$$

$$\Rightarrow \boxed{x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}}$$

2 Issues: • If $\Delta = a_1^2 - 4a_0 a_2 < 0$ "discriminant", no solution in $\mathbb{Q}$

- Even if $\Delta > 0$, there may not be a solution in $\mathbb{Q}$.

<u>E.g.:</u> $x^2 = 2 \Rightarrow x = \sqrt{2}$

<u>Claim</u>: $\sqrt{2} \notin \mathbb{Q}$

<u>Proof</u>: Suppose $\sqrt{2} = \frac{n}{m}$ with $n, m \in \mathbb{N}$

$$\Rightarrow m\sqrt{2} = n$$

$$\Rightarrow m^2 \sqrt{2}^2 = n^2$$

$$\Rightarrow m^2\, 2 = n^2$$

all prime factors must appear an even number of times

"odd" prime factor, contradiction!

This means we need more numbers, the irrational numbers $\mathbb{I}$.

<u>Fact</u>: if $a \in \mathbb{Q}$, then it can be written as a decimal expansion which

- either terminates, e.g. 2.351
- or becomes eventually periodic, e.g. $77.3515151\ldots = 77.3\overline{51}$