Recall: $f : (a,b) \to \mathbb{R}$ is continuous at $x_0 \in (a,b)$ if

$$\lim_{x \to x_0} f(x) = f(x_0) \qquad \text{(i.e., the limit exists)}$$

(If f is continuous at every point of its domain, we say "f is continuous".

Note: $f(x) = \frac{1}{x}$ $\qquad$ Domain$(f) = \mathbb{R} \setminus \{0\}$

f is continuous

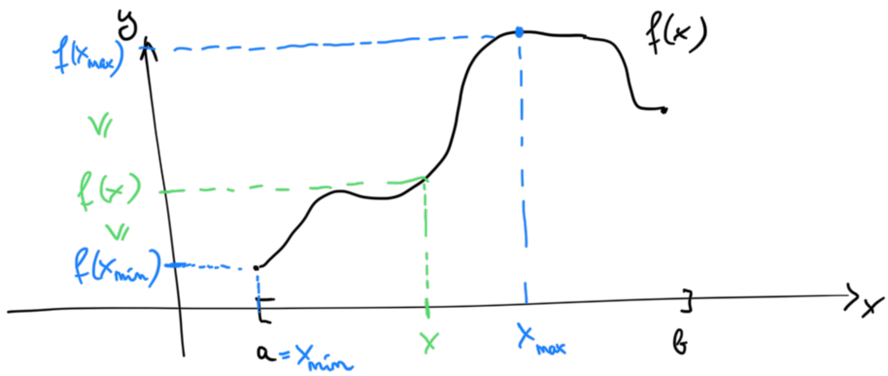
(but f is not cont. at $x=0$ because this point is not even in the domain)

Theorem: $f : [a,b] \to \mathbb{R}$ cont. (closed interval of definition),

then f assumes its minimum and maximum.

I.e. there exist $x_{min} \in [a,b]$ s.t. $f(x) \geq f(x_{min})$ $\forall x \in [a,b]$

$x_{max} \in [a,b]$ " $f(x) \leq f(x_{max})$ $\forall x \in [a,b]$



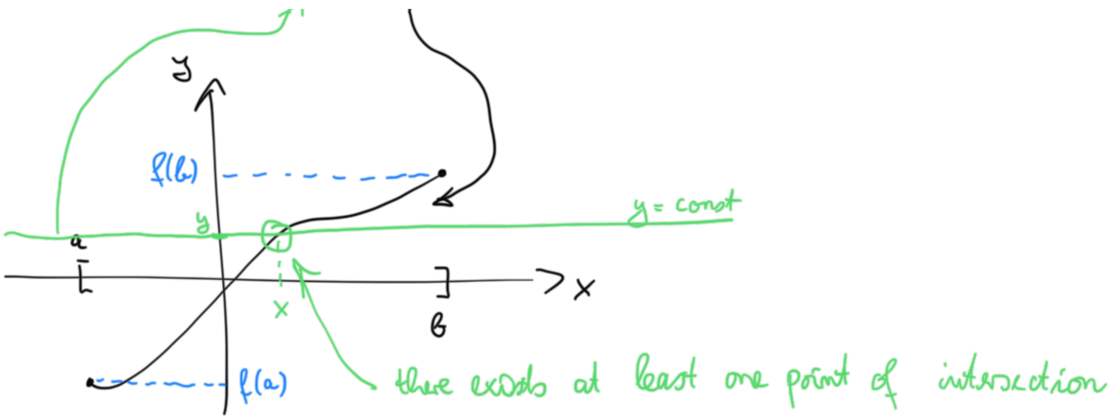
Intermediate value theorem:

$f : [a,b] \to \mathbb{R}$ cont.

Then f takes all intermediate values between $f(a)$ and $f(b)$:

For every $y \in [f(a), f(b)]$ (or $[f(b), f(a)]$ if $f(b) < f(a)$)

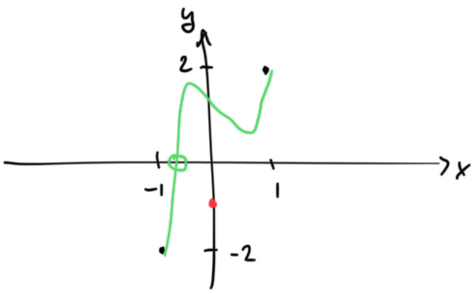
there is $x \in [a,b]$ s.t. $y = f(x)$.



Example: $p(x) = x^3 + x^2 + x - 1$

$p(1) = 1 + 1 + 1 - 1 = 2$

$p(-1) = -1 + 1 - 1 - 1 = -2$



So p has at least one root in the interval $[-1, 1]$.

Computational root finding: subdivide interval:

$p(0) = -1$

$\Rightarrow$ there is a root in $[0, 1]$

$p(\frac{1}{2}) = \frac{1}{8} + \frac{1}{4} + \frac{1}{2} - 1 < 0$

$\Rightarrow$ there is a root in $[\frac{1}{2}, 1]$

- - - - - -

From the limit laws:

f, g cont at x_0, then $f + g$ cont. at x_0
$f \cdot g$ " " x_0

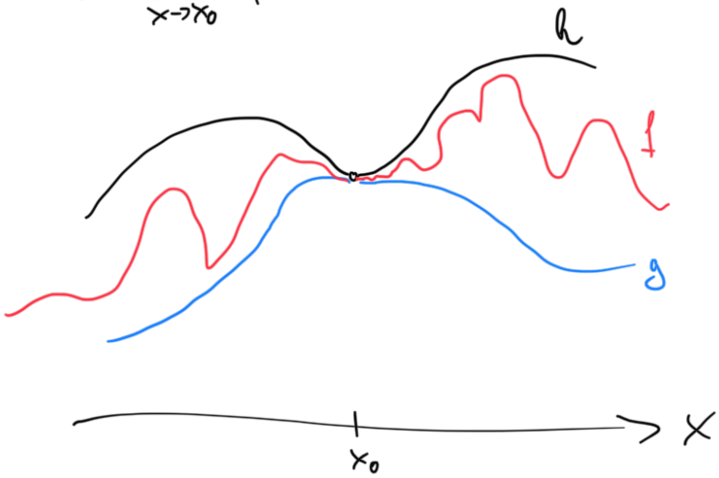
$\frac{f}{g}$ " " x_0 if $g(x_0) \neq 0$

$f(g(x)) = (f \circ g)(x)$ is cont. if f cont. at $g(x_0)$
g " " x_0

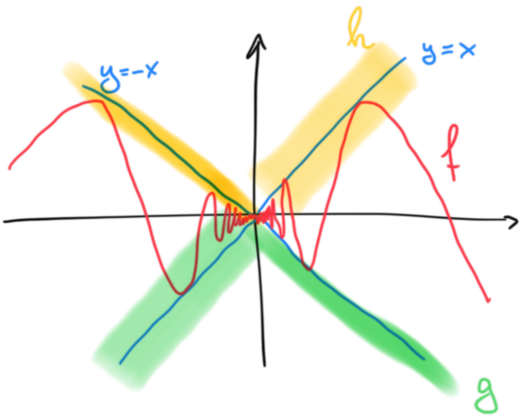
<u>Squeeze law</u>: $g(x) \leq f(x) \leq h(x)$ defined on (a,b) except perhaps at $x_0 \in (a,b)$

$$\lim_{x \to x_0} g(x) = L = \lim_{x \to x_0} h(x)$$

$$\Longrightarrow \lim_{x \to x_0} f(x) = L$$



<u>Examples</u>: ① $f(x) = x \sin \frac{1}{x}$ Domain$(f) = \mathbb{R} \setminus \{0\}$

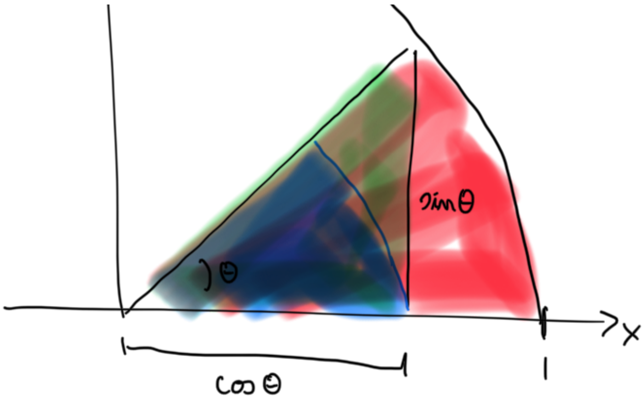


$$|f(x)| \leq |x| \underbrace{\left|\sin \frac{1}{x}\right|}_{\leq 1}$$

$$\leq |x|$$

$$\Rightarrow \lim_{x \to 0} f(x) = 0$$

② $f(\theta) = \dfrac{\sin \theta}{\theta}$



$A_1 = \frac{1}{2} \underbrace{\cos^2\theta}_{\text{radius}}\, \theta$ $\qquad$ $A_2 = \frac{1}{2}\cos\theta \sin\theta$ $\qquad$ $A_3 = \frac{1}{2}\theta$

$$A_1 \le A_2 \le A_3$$

$$\Rightarrow \quad \cos^2\theta\ \theta \le \cos\theta \sin\theta \le \theta$$

$$\Rightarrow \quad \cos\theta \le \frac{\sin\theta}{\theta} \le \frac{1}{\cos\theta}$$

$\cos\theta \xrightarrow{\theta\to 0} 1$ $\qquad$ $\frac{1}{\cos\theta} \xrightarrow{\theta\to 0} 1$

$$\Rightarrow \quad \boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$$

"Basic trigonometric limit"