Typo on homework:

show that $\exp(x+y) = \exp(x)\cdot\exp(y)$

$\exp(x) = e^x$ for some $e \in \mathbb{R}$

Look back at the definition: $\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

Put $x=1$: $e = e^1 = \exp(1) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

(For Analysis I: This limit exists, and is approximately equal to 2.7....)

Exponential function and natural logarithm

For $n \in \mathbb{N}$, $a^n = \underbrace{a\cdot a\cdot \ldots \cdot a}_{n \text{ factors}}$

note: $a^{n+m} = a^n a^m$ for $n, m \in \mathbb{N}$ $(*)$ $\Rightarrow (a^n)^m = a^{n\cdot m}$

Goal: extend this to more general sets of numbers while keeping the characteristic equation $(*)$ valid.

$m = -n$

- $n \in \mathbb{Z}$, $1 = a^0 = a^{\overbrace{n-n}^{+(-n)}} \overset{(*)}{=} a^n a^{-n}$

$\Rightarrow \boxed{a^{-n} = \frac{1}{a^n}}$

- $r = \frac{p}{q} \in \mathbb{Q}$: $a = a^{\frac{1}{q}q} \overset{(*)}{=} \left(a^{\frac{1}{q}}\right)^q$ $q \in \mathbb{N}$

$\Rightarrow a^{\frac{1}{q}} = \sqrt[q]{a}$

$\Rightarrow \boxed{a^r = \left(\sqrt[q]{a}\right)^p}$

From homework: $e^{x+y} = e^x e^y$ where (as above) $e^x = \exp(x)$

$\Rightarrow (e^x)^y = e^{xy}$ (2nd version of characteristic equation)

$\Rightarrow e^{xy} = (e^y)^x$

$= a^x$ with $a = e^y$

Fact: $e^y : \mathbb{R} \to (0, \infty)$ is invertible, define the inverse by $\ln$ "natural logarithm"

$\Rightarrow y = \ln a$

$\Rightarrow \boxed{a^x = e^{x \ln a}}$

Conclusion: every exponential function can be expressed by the "natural" exponential function exp, simply by rescaling x.

Limits involving the exponential function:

Q: what is $\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^x}$ $\qquad e^x \longrightarrow \infty$ as $x \to \infty$

Claim: $\frac{n}{e^n} \leq \left(\frac{2}{e}\right)^n$

Proof by induction: $n=1$: $\frac{1}{e^1} \leq \left(\frac{2}{e}\right)^1$ ✓

$n \to n+1$: $\frac{n+1}{e^{n+1}} = \frac{n+1}{e\, e^n} = \frac{n}{e\, e^n} + \frac{\overset{\leq n}{1}}{e\, e^n}$

$\leq \frac{n}{e\, e^n} + \frac{n}{e e^n} = \frac{2}{e} \frac{n}{e^n} \leq \left(\frac{2}{e}\right)^{n+1}$

$\frac{n}{e^n} \leq \left(\frac{2}{e}\right)^n$ by induction hypothesis

$$\frac{2}{e} < 1 \Rightarrow \overbrace{\lim_{n\to\infty} \frac{n}{e^n}}^{0} \leq \lim_{n\to\infty} \left(\frac{2}{e}\right)^n = 0 \qquad \text{squeeze law}$$

This shows the limit for $n \in \mathbb{N}$, for $x \in \mathbb{R}$, the argument is easily modified (round to nearest integer, this will not change much as $x \to \infty$.)

From here, it is easy to show that

$$\boxed{\lim_{x\to\infty} x^\alpha e^{-x} = 0 \quad \text{for every } \alpha > 0}$$

remember and use "as is"

" e^x grows faster than every power of x "

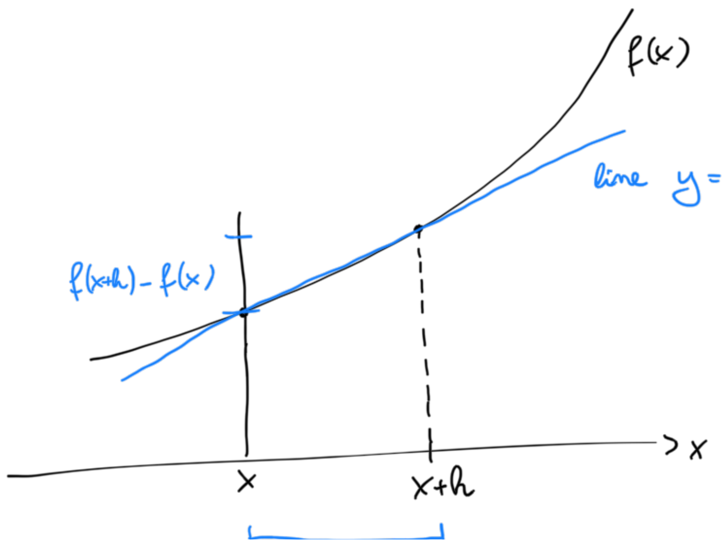
Inverse result:

$$\boxed{\lim_{x\to 0} x^\alpha \ln x = 0 \quad \text{for every } \alpha > 0}$$

remember and use "as is"

" $\ln x$ grows slower than every power of x "

Derivatives



line $y = f(x) + \underbrace{\frac{f(x+h) - f(x)}{h}}_{m(h)} h$

h

If h is small, the blue line can be seen as a linear approximation to $f(x)$.

<u>Idea:</u> Let $h \to 0$, expect that $m(h)$ converges, and the limit can be be interpreted as the <u>slope of f</u> at the point x.

<u>Def.:</u> The <u>derivative</u> of $f:(a,b) \longrightarrow \mathbb{R}$ at a point $x \in (a,b)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

<u>Examples:</u> $f(x) = \sin x$

$$f'(x) = \lim_{a \to 0} \frac{\overbrace{\sin(x+h)}^{(\sin x)(\cos h) + (\cos x)(\sin h)} - \sin x}{h}$$

$$= \lim_{h \to 0} \left(\sin x \underbrace{\frac{\cos h - 1}{h}}_{\xrightarrow{(*)} 0} + \cos x \underbrace{\frac{\sin h}{h}}_{} \right)$$

$$\xrightarrow[h \to 0]{} \cos x \quad \text{because} \quad \boxed{\lim_{h \to 0} \frac{\sin h}{h} = 1}$$

<u>Proof of (*):</u> Look at $g(h) = \dfrac{1 - \cos h}{h^2}$

$1 - \cos h = 2 \sin^2 \frac{h}{2}$

$\frac{h}{2} = y \Rightarrow h = 2y$

$$= 2 \frac{\sin^2 \frac{h}{2}}{h^2}$$

$$= 2 \frac{\sin^2 y}{(2y)^2}$$

$$= \frac{1}{2} \underbrace{\left(\frac{\sin y}{y} \right)^2}_{\to 1 \text{ as } y \to 0 \text{ or } h \to 0} \longrightarrow \frac{1}{2}$$

1 - cos h

If $\lim_{h \to 0} \frac{\qquad}{h^2} = 2$

then $\lim_{h \to 0} \frac{1-\cos h}{h} = 0$ (*)

Conclusion: $\boxed{\sin'(x) = \cos x}$

$\boxed{\cos'(x) = -\sin x}$ (by a similar argument).

Addition rule: if f', g' exist, then $(f+g)' = f' + g'$

Product rule: If f', g' exist, then $(fg)' = fg' + gf'$

Proof: $(fg)'(x) = \lim_{h \to 0} \frac{(fg)(x+h) - (fg)(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)}{h}$$

$$= \underbrace{\lim_{h \to 0} f(x+h) \frac{g(x+h)-g(x)}{h}}_{\underbrace{\lim_{h\to 0} f(x+h)}_{f(x)\ (*)}\ \underbrace{\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}}_{g'(x)}} + g(x) \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{f'(x)}$$

(*) because f is continuous.

Note: If the derivative of f exists at $x \in (a,b)$, then f is continuous at x.

"differentiability implies continuity" (but not the other way round.)