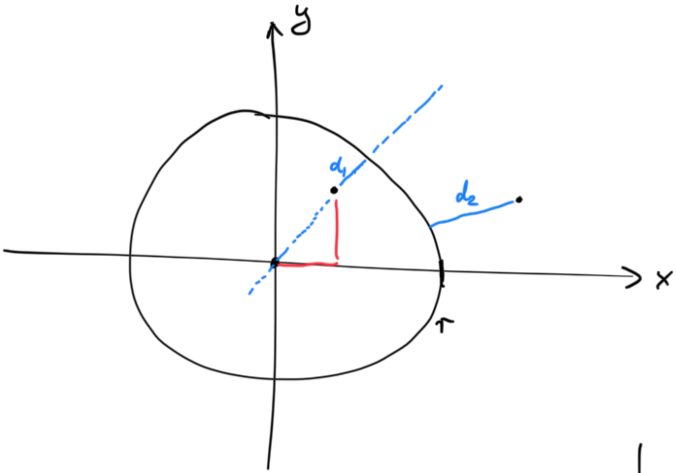
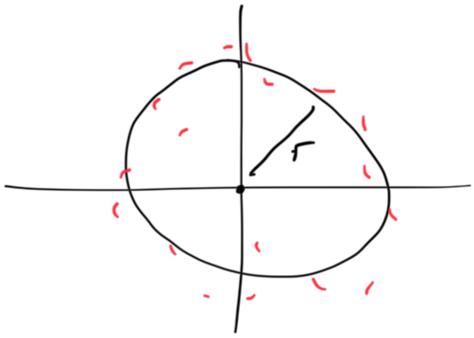
HW 6 #5:



Minimize $d_1^2(r) + d_2^2(r)$



Antiderivatives

Let f, F be functions. Then F is an antiderivative of f on some interval I if $F' = f$ on I.

E.g.: $f(x) = x^3$

$F(x) = \frac{1}{4}x^4$ is an anti-derivative of f, and so is $F(x) = \frac{1}{4}x^4 + 5$

Theorem: F is antiderivative of f, then G is an antiderivative of f

$\Longleftrightarrow$ $G = F + c$, $c \in \mathbb{R}$

Proof: "$\Leftarrow$" $G' = (F+C)' = F' + c' = f + 0 = f$

so G is an anti-derivative of f.

"$\Rightarrow$" Set $H = G - F$

$\Rightarrow H' = G' - F' = f - f = 0$

So H is a constant. Why?

$$\frac{H(y)-H(x)}{y-x} = \underbrace{H'(\xi)}_{=0} \quad , \xi \in (x,y) \qquad \text{MVT}$$

$$\Rightarrow H(y) = H(x) \quad \text{if } x \neq y$$

$$\Rightarrow H = c, \quad c \in \mathbb{R}$$

□

We write:

$$\int f(x)\,dx = F(x) + C$$

c is called "constant of integration"

<u>Examples:</u> 1) $\frac{d}{dx} x^{n+1} = (n+1)\, x^n \qquad n \neq -1, \; n \in \mathbb{R}$

$$\Rightarrow \boxed{\int x^n\,dx = \frac{1}{n+1} x^{n+1} + c \qquad n \neq -1}$$

2) $\frac{d}{dx} \ln x = \frac{1}{x} \qquad x > 0$

$$\Rightarrow \boxed{\int \frac{1}{x}\,dx = \ln x + c \qquad x > 0}$$

Note:

$$\frac{d}{dx} \ln(-x) = -\frac{1}{-x} = \frac{1}{x} \qquad x < 0$$

So we could write that

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

on any interval not including 0.

3) $\frac{d}{dx} \cos x = -\sin x$

$$\Rightarrow \int \sin x\,dx = -\cos x + c$$

4) $\int e^x\,dx = e^x + C$

5) $\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\ldots} = \frac{\cos^2 x - (-\sin x)\sin x}{\ldots} = \frac{1}{\ldots}$

$\frac{d}{dx} \quad \frac{d}{dx} \quad \cos x \qquad \cos^2 x \qquad \cos^2 x$

$$\Rightarrow \int \frac{1}{\cos^2 x}\, dx = \tan x + C$$

⑥ Recall: $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

$$\Rightarrow \int \frac{1}{1+x^2}\, dx = \arctan x + C$$

Translate key rules of differentiation into rules of integration:

Chain rule $\longrightarrow$ "integration by substitution"

$$\frac{d}{dx} F(u(x)) = F'(u(x))\; u'(x) \qquad\qquad F' = f$$
$$= f(u(x))\; u'(x)$$

$$\Rightarrow \int f(u(x))\; u'(x)\, dx = \int \frac{d}{dx} F(u(x))\, dx = F(u(x)) + C$$

Useful mnemonic (not rigorous, but useful and correct):

write $u'(x) = \frac{du}{dx}$ " $\Rightarrow$ " $du = u'(x)\, dx$

then $\int f(u(x))\; u'(x)\, dx = \int f(u)\, du = F(u(x)) + C$

Examples:

① $\int e^x \sin(e^x)\, dx$

$= \int \sin u\, du = -\cos u + C$

$= -\cos(e^x) + C$

$u(x) = e^x$

$\frac{du}{dx} = e^x \Rightarrow du = e^x\, dx$

② $\int \sqrt{1+x^2}\; \underbrace{x^3}_{x^2 \cdot x}\, dx$

$= \int \sqrt{u}\; \underbrace{x^2}_{= u-1}\; \frac{du}{2}$

$u = 1+x^2 \Rightarrow x^2 = u-1$

$\frac{du}{dx} = 2x \Rightarrow x\,dx = \frac{du}{2}$

$\int u^{\frac{3}{2}} \quad u^{\frac{1}{2}} \quad \frac{1}{\cancel{2}} u^{\frac{5}{2}} \quad \frac{1}{\cancel{2}} u^{\frac{3}{2}} + C$

$$= \frac{1}{2}\int (u^{\frac{3}{2}} - u^{\frac{1}{2}})\, du = \frac{\cancel{2}}{5} u^{\frac{5}{2}} - \frac{\cancel{2}}{3} u^{\frac{3}{2}} + C$$

$$= \frac{1}{5}(1+x^2)^{\frac{5}{2}} - \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c$$

③ $$\int \tan x \, dx = \int \frac{\sin x}{\cos x}\, dx$$

$u = \cos x \quad \Rightarrow \frac{du}{dx} = -\sin x$

$\Rightarrow du = -\sin x \, dx$

$$= -\int \frac{du}{u} = -\int \frac{1}{u}\, du$$

$$= -\ln|u| + c \qquad u \neq 0$$

$$= -\ln|\cos x| + c$$

Product rule ⟶ Integration by parts:

$$\frac{d}{dx}(u(x)\,v(x)) = u'(x)\,v(x) + u(x)\,v'(x)$$

$$\Rightarrow \int u'(x)\,v(x)\,dx = u(x)\,v(x) - \int u(x)\,v'(x)\,dx$$

Examples:

① $$\int x\,e^x\,dx = e^x\,x - \underbrace{\int e^x \cdot 1\,dx}_{e^x + c}$$

$$= (x-1)\,e^x + c$$