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<u>Select problems</u> :

① $\lim_{x\to 0} \underbrace{\sin x \cos\frac{1}{x}}$ $\qquad -1 \leq \cos\frac{1}{x} \leq 1$

This function is squeezed between $-\sin x$ and $+\sin x$

$= 0 \qquad$ as $\lim_{x\to 0}(-\sin x) = \lim_{x\to 0}(+\sin x) = 0$

② $\phi(x) = x \ln|x|$

- can this function be cont. extended at $x=0$

$\lim_{x\to 0} x\ln|x| = 0$, so we can define $\phi(0)=0$

- Same question for ϕ'

$\phi'(x) = \ln|x| + x\frac{1}{|x|} = \ln|x| + 1$

As $x \searrow 0$, $\phi'(x) \to \infty$, so there is a vertical asymptote at $x=0$.

$\Rightarrow$ no cont. extension is possible.

③ Curve sketching

$f(x) = x^2 \ln x$

- $D(f) = (0,\infty)$
- horizontal asymptotes: $\lim_{x\to\infty} f(x) = \infty$, no hor. asymptote
- vertical asymptotes: $\lim_{x\to 0} f(x) = 0$, no vert. asymptote

Look at

$f'(x) = 2x\ln x + x^2\frac{1}{x} = \underbrace{x}_{\geq 0}(2\ln x + 1)$

the only critical point in $D(f)$ is when $2\ln x + 1 = 0$

$$\Rightarrow \ln x = -\frac{1}{2}$$

$$\Rightarrow x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

at crit. point, f' changes sign from $-$ to $+$, so f has a min at

$$y = f\left(\frac{1}{\sqrt{e}}\right) = \left(\frac{1}{\sqrt{e}}\right)^2 \ln e^{-\frac{1}{2}} = \frac{2}{e}$$

Now look at

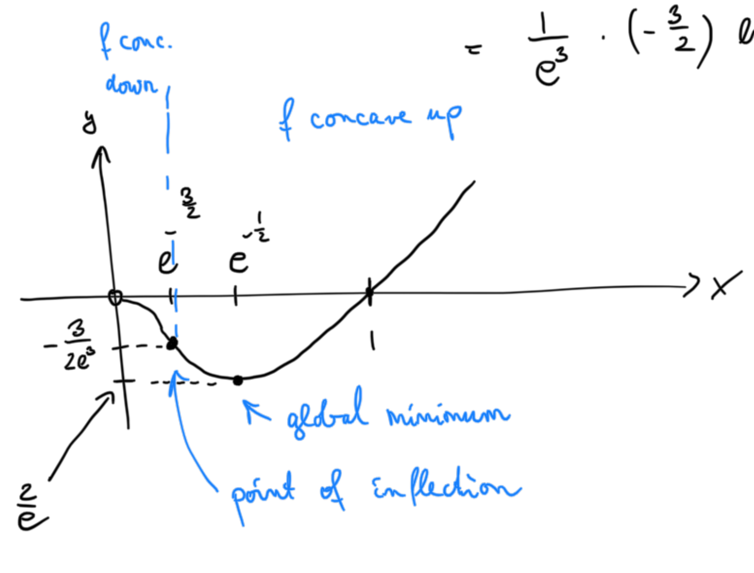
$$f''(x) = 2\ln x + 1 + x\,\frac{2}{x} = 2\ln x + 3$$

$$f''(x) = 0 \quad \text{if} \quad 2\ln x + 3 = 0 \Rightarrow \ln x = -\frac{3}{2} \Rightarrow x = e^{-\frac{3}{2}}$$

clearly, f'' changes from negative to positive at this point, so we have a point of inflection at

$$x = e^{-\frac{3}{2}}, \quad f(x) = \left(e^{-\frac{3}{2}}\right)^2 \ln e^{-\frac{3}{2}}$$

$$= \frac{1}{e^3}\cdot\left(-\frac{3}{2}\right)\ln e = -\frac{3}{2e^3}$$



(4) 800 passengers/day

€1.00 per ticket

For 5 cents increase/decrease of ticket price, 50 fewer/more passengers will travel.

Q: optimal ticket price.

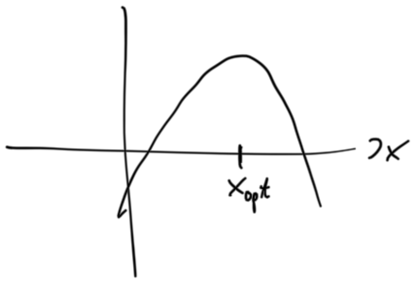
Let x be the units of decrease/increase of price

$P(x) = 1 + x \cdot \frac{5}{100}$ price

$N(x) = 800 - 50 \cdot x$ # of traveling passengers

Revenue $R(x) = P(x) \cdot N(x)$

$= \left(1 + \frac{5}{100}x\right)(800 - 50x)$



Can compute $R'(x) = 0$, solve for x, then plug into $P(x)$.

⑤ Find all points on the graph of $x^2 - xy + y^2 = 27$ where the tangent line is horizontal.

- Find points where $\frac{dy}{dx} = 0$
- Implicit differentiation: differentiate first, then solve

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x - y = 0 \quad \Rightarrow y = 2x$$

$$\frac{d}{dx}\left(y(x)\right)^2 = \underbrace{2y(x)}_{\text{outer der.}}\ \underbrace{\frac{dy}{dx}}_{\text{inner der.}}$$

$$\Rightarrow x^2 - x(2x) + (2x)^2 = 27$$

$$\Rightarrow \underbrace{x^2 - 2x^2 + 4x^2}_{3x^2} = 27 \quad \Rightarrow x^2 = 9$$

$$x = \pm 3$$

So the points are $(3, 6)$ and $(-3, -6)$

⑥ $\int \frac{\cos x}{\sqrt{4 - \sin^2 x}}\, dx$ $\quad 2u = \sin x \quad 2du = \cos x\, dx$

$$= \int \frac{\cos x}{\sqrt{4 - 4u^2}}\, dx = \int \frac{\cancel{2}\, du}{\cancel{2}\sqrt{1 - u^2}} = I$$

$\int \sqrt{4-4u}$ $\qquad\qquad$ $\int \cancel{2}\sqrt{1-u}$

This require a new trick: $\qquad U = \sin\Theta \qquad$ "trigonometric substitution"

$dU = \cos\Theta \, d\Theta \qquad \Theta = \arcsin U$

$$I = \int \frac{\cos\Theta \, d\Theta}{\sqrt{\underbrace{1-\sin^2\Theta}_{\cos^2\Theta}}} = \int d\Theta = \Theta + C_1$$

$$= \arcsin U + C_1$$

$$= \arcsin\left(\frac{\sin x}{2}\right) + C_1$$

⑦ $\displaystyle\int \underbrace{x^3 e^{-x^2} dx}_{\underbrace{x^2}_{-U} e^U \frac{dU}{-2}}$ $\qquad U = -x^2 \qquad dU = -2x\, dx$

then I.B.P.

$\displaystyle\int \frac{4x^2+x+1}{4x^3+x}\, dx \qquad$ partial fractions

$\displaystyle\int \frac{\sin x}{\cos x}\, dx \qquad U = \cos x$

$$\int x(\ln x)^3\, dx = \frac{1}{2}x^2(\ln x)^3 - \underbrace{\int \frac{1}{2}x^2\, 3(\ln x)^2\, \frac{1}{x}\, dx}_{\frac{3}{2}\int x(\ln x)^2\, dx}$$

Now keep going using the same pattern.

⑧ Compute the average of $\tan x$ over the interval $[-1,1]$.

As $\tan x$ is an odd function,

$$\int_{-a}^{a} \text{odd}\, dx = 0$$

Average of f over an interval [a, b]

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \overline{f}$$