Improper Integrals

$f : [a,b) \to \mathbb{R}$ integrable on any subinterval of $[a,b)$ that is closed,
b finite or $b = \infty$

$$\int_a^b f(x)\,dx = \lim_{r \to b} \int_a^r f(x)\,dx$$

if this limit exists, this is called an improper integral.

(Corresponding defn. for $f : (a,b] \to \mathbb{R}$ ...)

Examples:

- $$\int_1^\infty \frac{1}{x^\alpha}\,dx = \frac{1}{1-\alpha} x^{1-\alpha}\Big|_1^\infty = \lim_{r\to\infty} \frac{1}{1-\alpha}\left(r^{1-\alpha} - 1^{1-\alpha}\right) \qquad \alpha \neq 1$$

  $\Rightarrow$ when $1-\alpha < 0$, i.e. $\alpha > 1$, the integral converges to $\frac{1}{\alpha - 1}$.

  for $\alpha < 1$, the integral diverges.

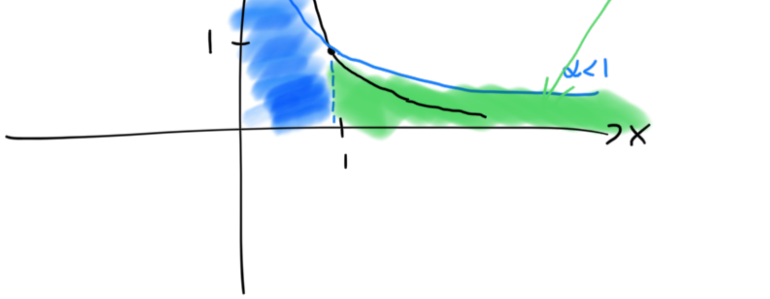
- $$\int_0^1 \frac{1}{x^\alpha}\,dx = \frac{1}{1-\alpha} x^{1-\alpha}\Big|_0^1 = \lim_{r \searrow 0} \frac{1}{1-\alpha}\left(1^{1-\alpha} - r^{1-\alpha}\right)$$

  $\Rightarrow$ when $1-\alpha > 0$, i.e. $\alpha < 1$, the integral converges to $\frac{1}{1-\alpha}$

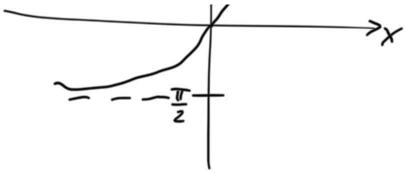
  for $\alpha > 1$, the integral diverges.

- $\alpha = 1$: $$\int_1^\infty \frac{1}{x}\,dx = \ln x\Big|_1^\infty = \lim_{r\to\infty} (\ln r - \ln 1) = \infty$$

  $$\int_0^1 \frac{1}{x}\,dx = \ln x\Big|_0^1 = \lim_{r \searrow 0} (\ln 1 - \ln r) = \infty$$



② $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \arctan x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$



③ For which values of p does the integral $\displaystyle\int_e^{\infty} \frac{dx}{x(\ln x)^p}$ converge?

$1 = \ln e^1$

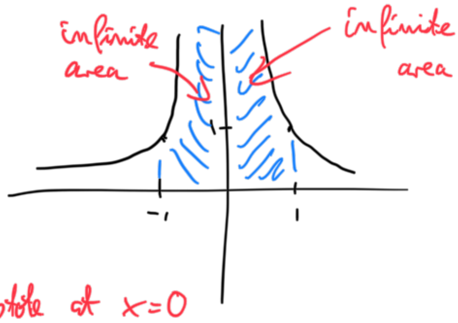
$u = \ln x \qquad du = \frac{1}{x}\,dx$

If $x \to \infty$, $u \to \infty$

$$\int_{x=e}^{\infty\ (x\to\infty)} \frac{dx}{x(\ln x)^p} = \int_{u=1}^{\infty\ (u\to\infty)} \frac{du}{u^p}$$

Back in case 1: converges for $p > 1$, diverges for $p \leq 1$.

④ $$\int_{-1}^{1} \frac{1}{x^2}\,dx \cancel{=} -x^{-1} \Big|_{-1}^{1} = -1 + (-1) = -2$$



Careful: this function has a vertical asymptote at $x=0$

Whenever this happens, we need to split the range of integration at the asymptote:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \underbrace{\int_{-1}^{0} \frac{1}{x^2}\,dx}_{=\infty} + \underbrace{\int_{0}^{1} \frac{1}{x^2}\,dx}_{=\infty}$$

by 1

⑤ The Gamma function:

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x}\,dx$$

$$\Gamma(1) = \int_0^{\infty} e^{-x}\,dx = -e^{-x}\Big|_0^{\infty} = -(0 - e^{-0}) = 1$$

$$\Gamma(n+1) = \int_0^{\infty} x^n e^{-x}\, dx = \underbrace{x^n(-e^{-x})\Big|_0^{\infty}}_{=0} - \int_0^{\infty} n x^{n-1}(-e^{-x})\, dx$$

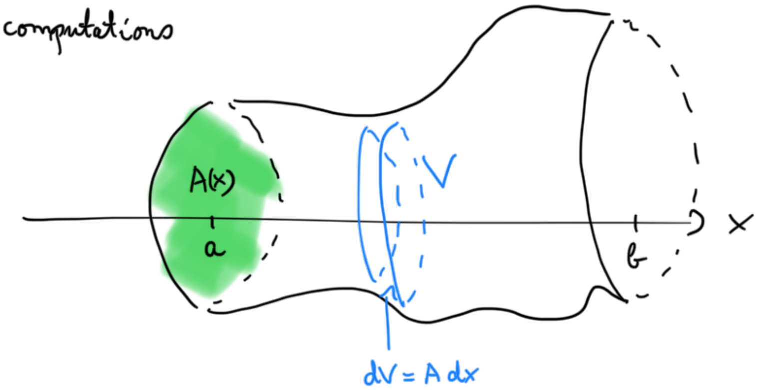
$$= n \cdot \Gamma(n)$$

$$\Rightarrow \Gamma(n) = (n-1)(n-2)\cdots 1 = (n-1)!$$ "(n-1) factorial"

Remark: Γ can be seen as a generalization of the factorial to non-integer values, shifted by 1.

Applications of the integral

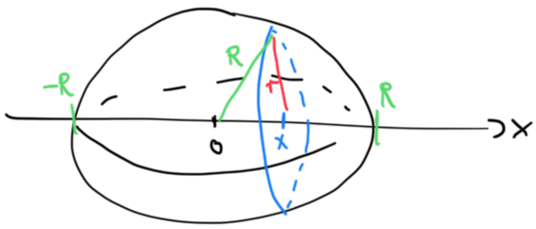
1. Volume computations



Q: what is the volume of such an object if $A(x)$ is known.

$$V = \int_a^b \underbrace{A(x)\, dx}_{dV}$$

E.g. volume of sphere from area of circle: πr^2



R: radius of sphere

r: radius of circle

$$x^2 + r^2 = R^2$$

$$V = \int_{-R}^{R} A(x)\, dx = \int_{-R}^{R} \pi r^2\, dx = \pi \int_{-R}^{R} (\underbrace{R^2 - x^2}_{r^2})\, dx$$

$$= \pi \left(R^2 x - \frac{1}{3} x^3 \right) \Big|_{-R}^{R} = 2\pi \left(R^2 x - \frac{1}{3} x^3 \right) \Big|_{0}^{R} = 2\pi \left(R^3 - \frac{1}{3} R^3 \right) = \frac{4}{3} \pi R^3$$

2. Work:

The work done by a force $F(x)$ moving an object from $x = a$ to $x = b$ is defined

$$W = \int_a^b F(x)\, dx \qquad \text{(in units: force times length)}$$