Calculus and Elements of Linear Algebra I

Final Exam

December 14, 2020

Some trigonometric identities:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sin 2x = 2\,\sin x\,\cos x$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \cos 2x = 2\,\cos^2 x - 1 = 1 - 2\,\sin^2 x$$

$$1 + \cot^2\theta = \csc^2\theta \qquad \tan 2x = \frac{2\,\tan x}{1 - \tan^2 x}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Useful integrals:

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C$$

$$\int \frac{du}{1+u^2} = \arctan u + C = -\operatorname{arccot} u + C'$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec}|u| + C$$

$$\int \sec u\,du = \ln\Big|\sec u + \tan u\Big| + C$$

1. Compute the limits

   (a) $\lim_{t\to\infty} e^{-t}\,\cos(t)\,\sin(t)$

   (b) $\lim_{\theta\to 0} \frac{1-\cos(2\theta)}{\theta^2}$

   (c) Compute the derivative of

   $$f(x) = \frac{1}{x}$$

   directly from its definition as the limit of a difference quotient.

(5+5+5)

2. Consider the function

$$f(x) = \frac{2 - x^2}{1 - x^2}.$$

What is the domain of f? Find horizontal and vertical asymptotes, local minima, local maxima, and inflection points of f. Identify the regions where the graph of f is concave upward or concave downward. Finally, sketch the graph into the coordinate system provided. (15)

3. A farmer owns a 10 km long stretch of land between two parallel rivers that are 2 km apart. What is the area of the largest rectangular corral he can enclose with

   (a) 2 km of fencing,

   (b) 5 km of fencing,

   assuming that no fence is needed along the river? (7+3)

4. Use implicit differentiation to find an equation for the tangent line to the graph of $\sin(x + y) = y^2 \cos(x)$ at point $(0, 0)$. (5)

5. Compute the following integrals:

   (a) $\displaystyle\int_0^1 \ln x \, dx$

   (b) $\displaystyle\int \frac{x^2 + 1}{x^2 - 1} \, dx$

   (5+10)

6. Is the following improper integral convergent? There is no need to compute the answer, but you should give detailed reasoning.

$$\int_0^\infty \frac{\ln x + e^{-x}}{1 + x^2} \, dx$$

(10)

7. Consider the differential equation

$$\frac{dy}{dt} = t^3 y^3.$$

   (a) Solve the initial value problem with $y(0) = 2$.

   (b) Does this equation have equilibrium points? Are they stable or unstable?

   (10+5)

8. Show that, for $u, v \in \mathbb{R}^3$,

$$||\mathbf{u}||^2 \, ||\mathbf{v}||^2 = (\mathbf{u} \cdot \mathbf{v})^2 + ||\mathbf{u} \times \mathbf{v}||^2 .$$

(5)

9. Find the general solution to the system of linear equations $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 2 & 0 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & -1 & 2 & 3 \\ 1 & 1 & 1 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -2 \\ -2 \\ 0 \\ -3 \end{pmatrix} .$$

(10)

10. Let $L \colon \mathbb{R}^4 \to \mathbb{R}^4$ be the "shift mapping" defined as follows:

$$L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} .$$

(a) Show that L is a linear transformation on $\mathbb{R}^4$.

(b) Write out the matrix S which represents L with respect to the standard basis.

(c) Find a basis for $\operatorname{Range} S$ and $\operatorname{Ker} S$.

(d) State the "rank-nullity theorem" and verify explicitly that the result obtained in part (c) matches the statement of the theorem.

(5+5+5+5)