HW1 Solutions

1(a). Quadratic formula:

$$z_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4\cdot 13}}{2} = \frac{-4 \pm \sqrt{-36}}{2}$$

$$= -2 \pm 3i$$

(b) Look at the discriminant $\Delta = \lambda^2 - 4\cdot 2\cdot \lambda$

$$= \lambda(\lambda - 8)$$

$\Delta = 0$ for $\lambda = 0$ or $\lambda = 8$ (one real solution)

$\Delta > 0$ for $\lambda > 8$ or $\lambda < 0$ (two real solutions)

$\Delta < 0$ for $\lambda \in (0, 8)$ (pair of complex-conjugate roots)

2. Use repeated long division:

$$(x^6 - 4x^5 - x^4 + 18x^3 - 18x^2 - 8x + 24) : (x-3) = x^5 - x^4 - 4x^3 + 6x^2 - 8$$

$$\begin{array}{l}
-\underline{|\, x^6 - 3x^5} \\
\quad -x^5 - x^4 \\
\quad -\underline{|\, -x^5 + 3x^4} \\
\qquad -4x^4 + 18x^3 \\
\qquad -\underline{|\, -4x^4 + 12x^3} \\
\qquad\quad 6x^3 - 18x^2 \\
\qquad\quad -\underline{|\, 6x^3 - 18x^2} \\
\qquad\qquad 0 - 8x + 24 \\
\qquad\qquad -\underline{|\, -8x + 24} \\
\qquad\qquad\qquad 0
\end{array}$$

By inspection, $x = -1$ is a root of the quotient polynomial. Divide it out:

$$(x^5 - x^4 - 4x^3 + 6x^2 - 8) : (x+1) = x^4 - 2x^3 - 2x^2 + 8x - 8$$

$$\begin{array}{l}
-\underline{|\, x^5 + x^4} \\
\quad -2x^4 - 4x^3
\end{array}$$

$$-\rfloor\; -2x^4-2x^3$$
$$-2x^3+6x^2$$
$$-\rfloor\; -2x^3-2x^2$$
$$8x^2$$
$$-\rfloor\; 8x^2+8x$$
$$-8x-8$$
$$-\rfloor\; -8x-8$$
$$0$$

By inspection, $x=2$ is a root of the quotient polynomial. Divide it out:

$$(x^4-2x^3-2x^2+8x-8):(x-2) = x^3-2x+4$$
$$-\rfloor\; x^4-2x^3$$
$$0\; -2x^2$$
$$-\rfloor\; -2x^2+4x$$
$$4x-8$$
$$-\rfloor\; 4x-8$$
$$0$$

By inspection, $x=-2$ is a root of the quotient, divide it out:

$$(x^3-2x+4):(x+2) = x^2-2x+2$$
$$-\rfloor\; x^3+2x^2$$
$$-2x^2-2x$$
$$-\rfloor\; -2x^2-4x$$
$$2x+4$$
$$-\rfloor\; 2x+4$$
$$0$$

By the quadratic formula, the remaining roots are

$$x = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Altogether, we have obtained the factorization

$$p(x) = (x-3)(x+1)(x-2)(x+2)(x-1-i)(x-1+i)$$

$$a^2-b^2-2abi$$

3(a). $$\frac{1}{z^2} = \frac{1}{a^2+2abi-b^2} = \frac{\quad}{\underbrace{(a^2-b^2+2abi)(a^2-b^2-2abi)}}$$

$$= (a^2-b^2)^2 + (2ab)^2$$
$$= a^4 - 2a^2b^2 + b^4 + 4a^2b^2$$
$$= (a^2+b^2)^2$$

$$= \frac{a^2-b^2}{(a^2+b^2)^2} + i\,\frac{-2ab}{(a^2+b^2)^2}$$

(b) $$\frac{z+1}{2z-5} = \frac{a+ib+1}{2a+2ib-5} = \frac{(a+ib+1)(2a-5-2ib)}{(2a-5+2ib)(2a-5-2ib)}$$

$$= \frac{(a+1)(2a-5)+2b^2+ib(2a-5)-i2b(a+1)}{(2a-5)^2+(2b)^2}$$

$$= \frac{(a+1)(2a-5)+2b^2}{(2a-5)^2+4b^2} + i\,\frac{-7b}{(2a-5)^2+4b^2}$$

It is possible to multiply out some of these terms, but this does not really simplify the expression further.

(c) $$z^3 = (a+ib)^3 = a^3 + 3a^2\,ib + 3a(ib)^2 + (ib)^3$$

$$= a^3 - 3ab^2 + i(3a^2b - b^3)$$

4(a) $$\left|\frac{1+i}{2-i}\right|^2 = \frac{(1+i)(1-i)}{(2-i)(2+i)} = \frac{1+1}{4+1} = \frac{2}{5}$$

$$\Rightarrow \left|\frac{1+i}{2-i}\right| = \sqrt{\frac{2}{5}}$$

For (*): the roots are

$$x = \frac{44 \pm \sqrt{44^2 - 4\cdot 32\cdot 15}}{\quad}$$

(b) $|\ldots| > |x-2|$

(b) $|6-4x| \geq |x| - 1$

$$\Rightarrow (6-4x)^2 \geq (x-2)^2$$

$$\Rightarrow 36 - 48x + 16x^2 \geq x^2 - 4x + 4$$

$$\Rightarrow 15x^2 - 44x + 32 \geq 0$$

$$\Rightarrow (3x-4)(5x-8) \geq 0$$

$$\Rightarrow x \geq \frac{8}{5} \quad \text{or} \quad x \leq \frac{4}{3}$$

$$30$$

$$= \frac{44 \pm \sqrt{16}}{30}$$

$$= \frac{48}{30} \text{ or } \frac{40}{30}$$

$$= \frac{8}{5} \text{ or } \frac{4}{3}$$

5(a). Write $v = a+bi$, $w = c+di$

$$\Rightarrow v^* w^* = (a-bi)(c-di) = ac - bd - adi - cbi$$

$$= (ac - bd + adi + cbi)^*$$

$$= \left((a+bi)(c+di)\right)^* = (vw)^*$$

(b) For $n=1$, the statement is trivial.

Now assume that $(z^n)^* = (z^*)^n$ for some $n \in \mathbb{N}$. $(*)$

Then $(z^{n+1})^* = (z^n z)^*$

$= (z^n)^* z^*$ by part (a) with $v = z^n$, $w = z$

$= (z^*)^n z^*$ by $(*)$

$= z^{n+1}$

Thus, we conclude that the statement must be true for all $n \in \mathbb{N}$.

Note: This technique is called "proof by induction" and is quite common for combinatorial-type problems.