Homework 11 Solutions

1. Augmented matrix:

$$\left(\underbrace{\begin{matrix}1 & 3 & -5\\ 1 & 4 & -8\\ -3 & -7 & 9\end{matrix}}_{A}\ \middle|\ \underbrace{\begin{matrix}4\\ 7\\ -6\end{matrix}}_{b}\right) \xrightarrow[R3+3R1 \to R3]{R2-R1 \to R2} \left(\begin{matrix}1 & 3 & -5\\ 0 & 1 & -3\\ 0 & 2 & -6\end{matrix}\ \middle|\ \begin{matrix}4\\ 3\\ 6\end{matrix}\right)$$

$$\xrightarrow[R3-2R2 \to R3]{R1-3R2 \to R1} \left(\begin{matrix}1 & 0 & 4\\ 0 & 1 & -3\\ 0 & 0 & 0\end{matrix}\ \middle|\ \begin{matrix}-5\\ 3\\ 0\end{matrix}\right)$$

$$\Rightarrow x = \begin{pmatrix}-5\\ 3\\ 0\end{pmatrix} + \lambda \begin{pmatrix}4\\ -3\\ -1\end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

Note: other representations may be correct, too!

Check: (not required, but useful)

$$A\begin{pmatrix}-5\\ 3\\ 0\end{pmatrix} = \begin{pmatrix}-5\cdot 1 + 3\cdot 3 + 0\cdot(-5)\\ -5\cdot 1 + 3\cdot 4 + 0\cdot(-8)\\ -5\cdot(-3) + 3\cdot(-7) + 0\cdot 9\end{pmatrix} = \begin{pmatrix}4\\ 7\\ -6\end{pmatrix} = b \quad \checkmark$$

$$A\begin{pmatrix}4\\ -3\\ -1\end{pmatrix} = \begin{pmatrix}4\cdot 1 + (-3)\cdot 3 + (-1)(-5)\\ 4\cdot 1 + (-3)\cdot 4 + (-1)\cdot(-8)\\ 4\cdot(-3) + (-3)\cdot(-7) + (-1)\cdot 9\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \quad \checkmark$$

2. Write out augmented matrix:

$$\left(\begin{matrix}1 & \alpha & 0\end{matrix}\ \middle|\ 1\right) \xrightarrow{R2 \to R1} \left(\begin{matrix}1 & -1 & 3\end{matrix}\ \middle|\ -1\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 2 & -2 & \alpha & -2 \end{array}\right) \xrightarrow[R3-2R2 \to R3]{R1-R2\to R2} \left(\begin{array}{ccc|c} 0 & \alpha+1 & -3 & 2 \\ 0 & 0 & \alpha-6 & 0 \end{array}\right)$$

This matrix has 3 pivots unless $\alpha=-1$ or $\alpha=6$, thus rank $A=3$ and the system has a unique solution.

Let's look at the exceptional cases:

- When $\alpha=-1$, the last two equations are

$$-3x_3 = 2 \quad \text{and} \quad -7x_3 = 0$$

$\Rightarrow$ the system is inconsistent

- When $\alpha=6$, we obtain the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 7 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ the solution is not unique.

Not required: in the last case, we can continue and compute

$$\xrightarrow[\frac{R2}{7} \to R2]{\frac{R2}{7}+R1 \to R1} \left(\begin{array}{ccc|c} 1 & 0 & \frac{18}{7} & -\frac{5}{7} \\ 0 & 1 & -\frac{3}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so the general solution is

$$x = \begin{pmatrix} -\frac{5}{7} \\ \frac{2}{7} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{18}{7} \\ -\frac{3}{7} \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

3. We need to solve

$$x_1 b_1 + x_2 b_2 + x_3 b_3 = v \quad \text{or} \quad Ax = v$$

with

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 3 \end{array}\right) \xrightarrow[R3-R2 \to R3]{\substack{R1 \to R2 \\ R2 \to R1}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{R2-R3}{2} \to R3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R2-R3 \to R2]{R1-R3 \to R1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\Rightarrow \quad x = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

<u>Check:</u> (not required) $\quad 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ✓

4. Write $v_1, \dots, v_4$ as columns into a matrix and start Gaussian elimination

$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & -1 & -1 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \xrightarrow[R4-R1 \to R4]{-R2 \to R2} \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -3 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow[R3+R4 \to R4]{R2+R3 \to R3} \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$0 \qquad 0 \qquad 0$$

$\Rightarrow$ all rows/columns have pivots, so the matrix has full rank.

$\Rightarrow$ $v_1, \dots, v_4$ are a basis of $\mathbb{R}^4$

5. Suppose that none of the matrices A, B is singular

$\Rightarrow$ $\ker A = \{0\}$

$\ker B = \{0\}$

Take any non-zero vector v

$\Rightarrow$ $w := Bv \neq 0$

$\Rightarrow$ $Aw \neq 0$

$\Rightarrow$ $0 \neq ABv = 0v = 0$, a contradiction.