Homework 5 Solutions

1(a) $x^2 y = x + 2$

$\Rightarrow 2xy + x^2 \frac{dy}{dx} = 1$

Plug in $(x,y) = (2,1)$ $\Rightarrow$ $2 \cdot 2 \cdot 1 + 2^2 \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = -\frac{3}{4}$

Thus, the tangent line is the line with slope $-\frac{3}{4}$ through $(2,1)$.

To find its y-intercept:

$y = -\frac{3}{4}x + b$

$\Rightarrow 1 = -\frac{3}{4} \cdot 2 + b \quad \Rightarrow b = 1 + \frac{3}{2} = \frac{5}{2}$

So the equation for the tangent line is $y = -\frac{3}{4}x + \frac{5}{2}$.

(b) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5$

$\Rightarrow \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dt} = 0 \quad \Rightarrow \quad x^{-\frac{1}{3}} + y^{-\frac{1}{3}}\frac{dy}{dx} = 0$

At point $(8,1)$: $\frac{1}{\sqrt[3]{8}} + \frac{1}{\sqrt[3]{1}}\frac{dy}{dx} = 0$

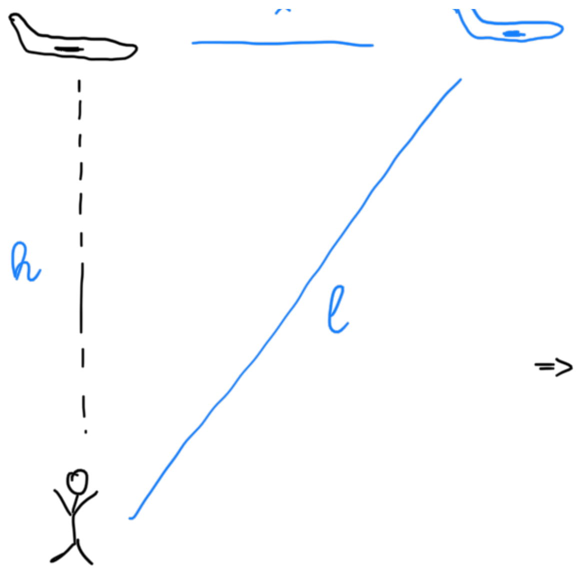
$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}$

Tangent line equation: $y = -\frac{1}{2}x + b$

$\Rightarrow 1 = -\frac{1}{2}8 + b \quad \Rightarrow b = 5$

So tangent line has equation $y = -\frac{1}{2}x + 5$

2.



$$l^2 = h^2 + x^2$$

$$\Rightarrow \cancel{2} l \frac{dl}{dt} = \cancel{2} x \frac{dx}{dt} \qquad (*)$$

Here x is distance traveled in 30s: $x = 300 \frac{m}{s} \cdot 30\ s$

$= 9\ km$

$$\Rightarrow l = \sqrt{(5\ km)^2 + (9\ km)^2} = \sqrt{25 + 81}\ km = \sqrt{106}\ km$$

$$\approx 10\ km$$

Using $(*)$:

$$10\ km \frac{dl}{dt} \approx 9\ km \cdot 300 \frac{m}{s}$$

$$\Rightarrow \frac{dl}{dt} \approx 270 \frac{m}{s}$$

3. Volume of sphere: $V = \frac{4}{3}\pi r^3$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Here: $\frac{dV}{dt} = 100\pi \frac{cm^3}{s}$, $r = 10\ cm$

$$\Rightarrow \cancel{100}\cancel{\pi} \frac{cm^{\cancel{3}}}{s} = 4\cancel{\pi}\ \cancel{10^2}\ \cancel{cm^2} \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{1}{4} \ \frac{\text{cm}}{\text{s}}$$

4. $\arcsin\left(\sin y\right) = y \qquad\qquad x = \sin y$

$$\Rightarrow \quad \arcsin'(x)\ \underbrace{\sin'(y)}_{=\cos y} = 1$$

$$\Rightarrow \quad \arcsin'(x) = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

5(a) $f(x) = x^3 - 3x + 3$

$f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$

$f'(x) = 0 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$

For $x < -1$, $f'(x) > 0$ so $f(x)$ is increasing

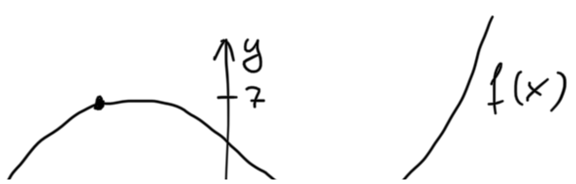
$x \in (-1,1)$, $f'(x) < 0$ so $f(x)$ is decreasing

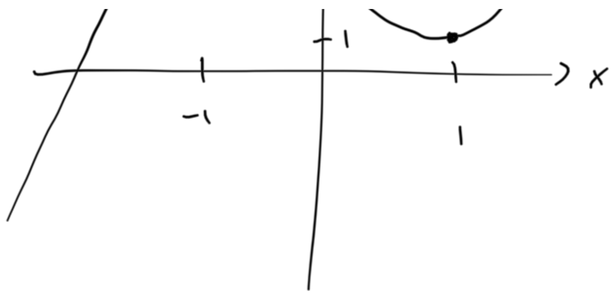
$x > 1$, $f'(x) > 0$ so $f(x)$ is increasing

$\Rightarrow$ f has a local maximum at $x = -1$,

a local minimum at $x = 1$

Sketch (not required):





(b) $f(x) = x^3 + 3x + 3$

$f'(x) = 3(x^2+1)$

$f'(x) = 0$ does not have any real roots $\Rightarrow$ no critical points

(c) $g(t) = \cos(\omega t)$, $\omega \neq 0$

$g'(t) = -\omega \sin \omega t$

$g'(t) = 0$ if $\omega t = n\pi$, $n \in \mathbb{Z}$

$\Rightarrow t = \frac{n\pi}{\omega}$

If n is even, then $\sin \omega t$ changes sign from $-$ to $+$,

so $g'(t)$ changes sign from $+$ to $-$,

$\Rightarrow \cos \omega t$ has a local maximum at $t = \frac{n\pi}{\omega}$, n even.

When n is odd, the signs are reversed, so that

$\cos \omega t$ has a local minimum at $t = \frac{n\pi}{\omega}$, n odd.