Homework 6 Solutions

1.

$$f(x) = \frac{x^2}{x^2-1} = \frac{x^2}{(x+1)(x-1)}$$

- $D(f) = \mathbb{R} \setminus \{-1, 1\}$
- $(0,0)$ is x-intercept and y-intercept, there are no further x-intercepts.
- $\lim_{x \to \pm\infty} f(x) = 1$, so $y = 1$ is the horizontal asymptote
- The vertical asymptotes are $x=-1$ and $x=1$, with

$$\lim_{x \nearrow -1} f(x) = \infty, \quad \lim_{x \searrow -1} f(x) = -\infty,$$

$$\lim_{x \nearrow 1} f(x) = -\infty, \quad \lim_{x \searrow 1} f(x) = \infty.$$

- $$f'(x) = \frac{2x(x^2-1) - 2x\, x^2}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$$

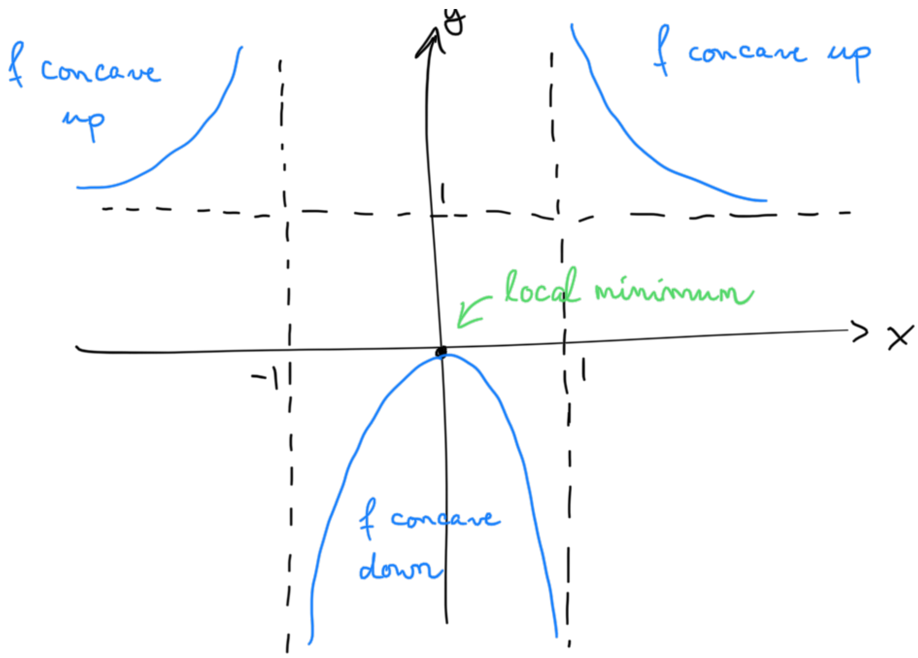
so f' changes sign from $+$ to $-$ at $x=0$, so there is a local maximum at $x=0$.

- $$f''(x) = \frac{-2(x^2-1)^2 - 2 \cdot 2x\,(x^2-1)(-2x)}{(x^2-1)^4}$$

$$= \frac{-2(x^2-1) + 8x^2}{(x^2-1)^3} = 2\,\frac{3x^2+1}{(x^2-1)^3}$$

$\Rightarrow f''(x) > 0 \Leftrightarrow x^2 - 1 > 0 \Rightarrow x > 1$ or $x < -1$

$f''(x) < 0$ if $x \in (-1, 1)$



2. $f(x) = \sqrt{x} - \ln x$

- $D(f) = (0, \infty)$
- no y-intercept as $x=0$ is not in $D(f)$

  x-intercept: the equation $\sqrt{x} = \ln x$ cannot be solved in a systematic way. We shall see that there is no solution in fact.

- $\lim\limits_{x \to 0} f(x) = \infty$, so $x=0$ is a vertical asymptote

  $\lim\limits_{x \to \infty} f(x) = \infty$, so no horizontal asymptote

- $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{x} = \frac{1}{2x}\left(\sqrt{x} - 2\right)$

  f' is changing sign from $-$ to $+$ at $\sqrt{x} = 2 \Rightarrow x = 4$

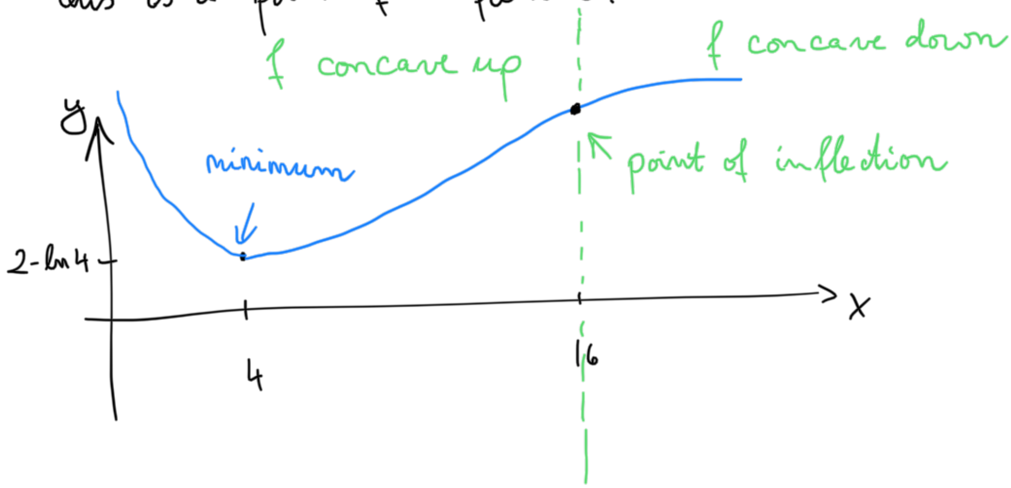
  This is the location of a local (and global) minimum.

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} + \frac{1}{x^2} = \frac{1}{x^2}\left(-\frac{\sqrt{x}}{4} + 1\right)$

- $f(x) = \frac{1}{2}(-\frac{1}{2})x \quad + x^2 \quad - x^2(\; 4 \quad + 1)$

f'' is changing sign from + to − at $\frac{\sqrt{x}}{4} = 1 \Rightarrow x = 16$

this is a point of inflection.



3. $f(x) = e^{-\frac{1}{x}}$

- $D(f) = \mathbb{R} \setminus \{0\}$
- no intercepts (note that $e^s > 0$ for any $s \in \mathbb{R}$!)
- $\lim\limits_{x \to -\infty} e^{-\frac{1}{x}} = \lim\limits_{t \searrow 0} e^t = 1$

$\lim\limits_{x \to \infty} e^{-\frac{1}{x}} = \lim\limits_{t \nearrow 0} e^t = 1$

so the horizontal asymptote is $y = 1$

- $\lim\limits_{x \nearrow 0} e^{-\frac{1}{x}} = \lim\limits_{t \to \infty} e^t = \infty$

$\lim\limits_{x \searrow 0} e^{-\frac{1}{x}} = \lim\limits_{t \to -\infty} e^t = 0$

so there is a "left-sided" vertical asymptote at $x=0$

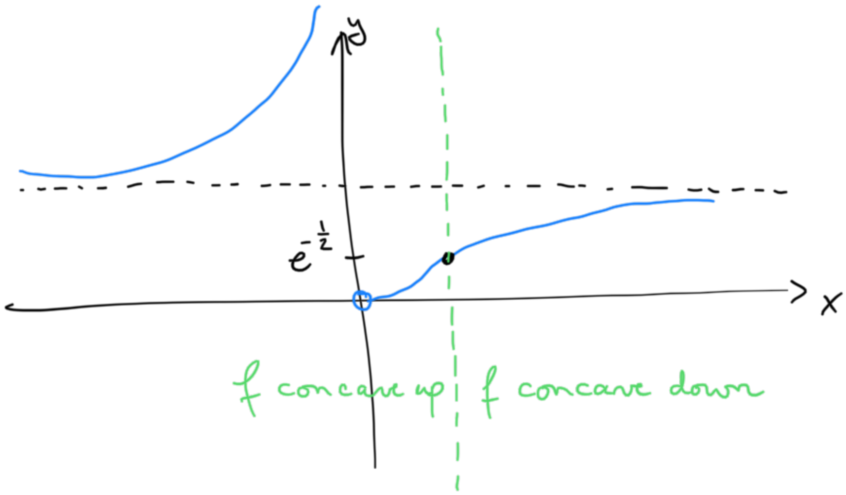
- $f'(x) = \frac{1}{x^2} e^{-\frac{1}{x}} > 0$

  $\Rightarrow$ f is increasing everywhere

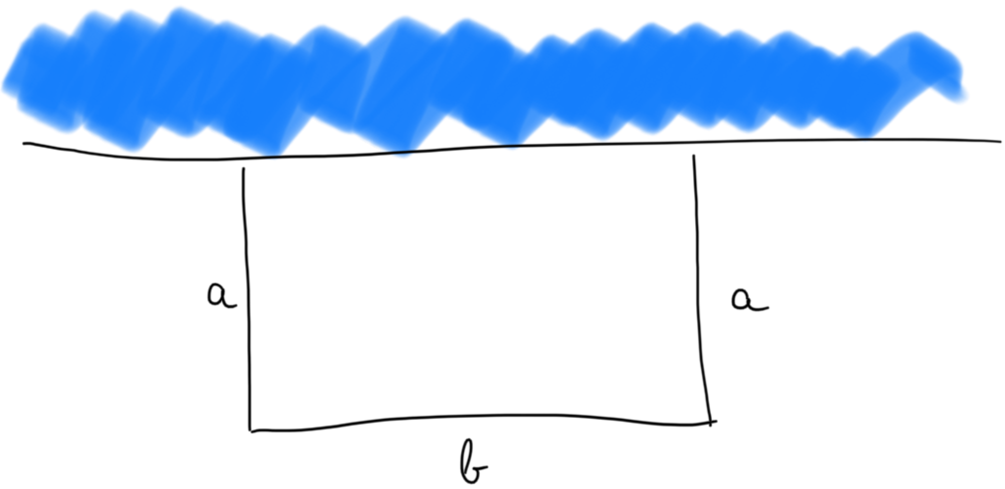
- $f''(x) = -2x^{-3} e^{-\frac{1}{x}} + \frac{1}{x^2} \frac{1}{x^2} e^{-\frac{1}{x}} = x^{-4} e^{-\frac{1}{x}} (1-2x)$

  for $x < \frac{1}{2}$, $f'' > 0$ $\Rightarrow$ f concave up

  $x > \frac{1}{2}$, $f'' < 0$ $\Rightarrow$ f concave down



4.



$A = ab$

$L = 2a + b$ (length of fence, given)

$\Rightarrow b = L - 2a \Rightarrow A(a) = a(L-2a) = aL - 2a^2$

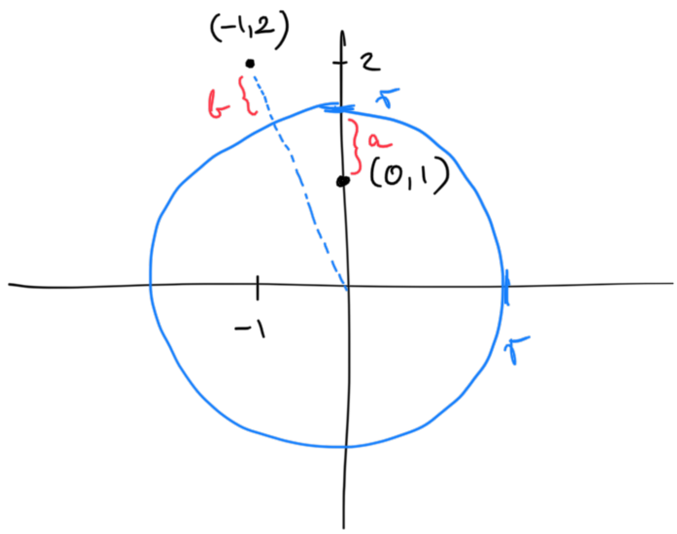
$A'(a) = L - 4a$

$A'(a) = 0 \Rightarrow L = 4a \Rightarrow a = \frac{L}{4} = \frac{500\,m}{4} = 125\,m$

Then $b = 500m - 250m = 250\,m$

Since $A = 0$ when $b = L$, this only critical point must correspond to a maximum of the area A.

5.



$a = r - 1$ $b = \sqrt{1^2 + 2^2} - r$

Thus, the task is to minimize

$$f(r) = (r-1) + (r-\sqrt{5})$$

$$= r^2 - 2r + 1 + r^2 - 2\sqrt{5}\,r + 5$$

$$= 2r^2 - 2(1+\sqrt{5})\,r + 6$$

$$f'(r) = 4r - 2(1+\sqrt{5}) \equiv 0 \implies r = \frac{1+\sqrt{5}}{2}$$

Since $f(r)$ has a positive leading coefficient, this critical point must correspond to a minimum of f.