Complex Analysis

Homework 1

Due in class Wednesday, September 22, 2021

1. Two fields K and L are isomorphic if there exists a bijective map $\phi\colon K \to L$ such that $\phi(x+y) = \phi(x) + \phi(y)$ and $\phi(xy) = \phi(x)\,\phi(y)$ for all $x, y \in K$.

   Show that $\mathbb{C}$ is isomorphic to the set of real 2×2 matrices of the form

   $$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

   endowed with the usual matrix addition and matrix multiplication.

2. Derive the Cauchy–Riemann equations by referring to the previous problem.

   Hint: Interpret $f\colon \Omega \subset \mathbb{C} \to \mathbb{C}$ as a map between subsets of $\mathbb{R}^2$ and compute its Jacobian. The CR-equations now follow by inspection. (Explain carefully why!)

3. Let $\Omega \subset \mathbb{C}$ be open and $f\colon \Omega \to \mathbb{C}$ be holomorphic. Set $\Omega^* = \{\overline{z}\colon z \in \Omega\}$ and define $g\colon \Omega^* \to \mathbb{C}$ by

   $$g(z) = \overline{f(\overline{z})}\,.$$

   Show that g is holomorphic with

   $$g'(z) = \overline{f'(\overline{z})}\,.$$

4. Let $\Omega \subset \mathbb{C}$ be open and $f\colon \Omega \to \mathbb{C}$ be holomorphic with $\operatorname{Re} f$ constant. Prove that f is a constant function.

5. Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n\, z^n$ when

   (a) $a_n = n^2$

   (b) $a_n = n!$

   (c) $a_n = (n!)^3/(3n)!$

   Hint: Use Stirling's formula, which says that

   $$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n$$

   as $n \to \infty$.