Complex Analysis

Homework 3

Due in class Wednesday, October 6, 2021

1. (Stein & Shakarchi, Chapter 2 Exercise 3.) Evaluate the integrals

$$\int_0^\infty \mathrm{e}^{-ax} \cos(bx) \,\mathrm{d}x \quad \text{and} \quad \int_0^\infty \mathrm{e}^{-ax} \sin(bx) \,\mathrm{d}x$$

with $a > 0$ by integrating e^{-Az}, $A = \sqrt{a^2 + b^2}$, over an appropriate sector with angle ω, with $\cos\omega = a/A$.

2. (Cf. Stein & Shakarchi, Chapter 2 Exercise 7.) Let $f\colon D_1(0) \to \mathbb{C}$ be holomorphic, and let

$$d = \sup_{w,z \in D_1(0)} |f(w) - f(z)|$$

denote the diameter of the image of z. Show that

$$2\,|f'(0)| \leq d$$

and that equality holds if and only if f is linear.

3. (Stein & Shakarchi, Chapter 2 Exercise 10.) Weierstrass's theorem states that a continuous function on $[0, 1]$ can be uniformly approximated by polynomials. Can every continuous function on the closed unit disc be approximated uniformly by polynomials in the variable z?

4. (Stein & Shakarchi, Chapter 2 Exercise 13.) Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is entire and such that for every $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n\,(z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

Hint: Use the fact that $c_n\,n! = f^{(n)}(z_0)$ and use a countability argument.