Complex Analysis

Homework 5

Due in class Thursday, October 21, 2021

1. (Stein & Shakarchi, Chapter 3 Exercise 7.) Prove that, for $a > 1$,

$$\int_0^{2\pi} \frac{1}{(a+\cos\theta)^2}\,\mathrm{d}\theta = 2\pi\,\frac{a}{(a^2-1)^{3/2}}\,.$$

   Hint: Recognize this expression as a parameterization of a line integral in the complex plane, then use the residue theorem.

2. (Stein & Shakarchi, Chapter 3 Exercise 9.) Show that

$$\int_0^1 \log(\sin\pi x)\,\mathrm{d}x = -\log 2\,.$$

   Hint: Integrate along the contour $\gamma = \mathrm{i}\,(\infty,0]) \cup [0,1] \cup 1+\mathrm{i}\,[0,\infty)$.

3. (Stein & Shakarchi, Chapter 3 Exercise 10.) Show that, for $a > 0$,

$$\int_0^\infty \frac{\log x}{a^2+x^2}\,\mathrm{d}x = \frac{\pi}{2a}\,\log a\,.$$

   Hint: Integrate along the contour bounded by the semicircle in the upper halfplane of radius r, the semicircle in the upper halfplane of radius R, both centered at the origin, plus the line segments along the real axis connecting the two semicircles. Then let $r \to 0$ and $R \to \infty$.

4. (Stein & Shakarchi, Chapter 3 Exercise 13.) Suppose f is holomorphic in a punctured disc $D_r(z_0) \setminus \{z_0\}$ and that

$$|f(z)| \le A\,|z-z_0|^{\varepsilon-1}$$

   for some $\varepsilon > 0$ and $A > 0$ and all z near z_0. Show that the singularity of f at z_0 is removable.