Complex Analysis

Homework 6

Due in class Thursday, October 28, 2021

1. (Stein & Shakarchi, Chapter 2 Exercise 8.)

   If f is a holomorphic function on the strip $-1 < y < 1$, $x \in \mathbb{R}$ with $|f(z)| \leq A\,(1+|z|)^\eta$, η a fixed real number, for all z in that strip, show that for each integer $n \geq 0$ there exists $A_n \geq 0$ so that, for all $x \in \mathbb{R}$,

   $$|f^{(n)}(x)| \leq A_n\,(1+|x|)^\eta\,.$$

   Hint: Cauchy inequalities.

2. (Stein & Shakarchi, Chapter 4 Exercise 2.) If $f \in \mathcal{F}_a$ with $a > 0$, then for any positive integer n one has $f^{(n)} \in \mathcal{F}_b$ whenever $0 \leq b < a$.

   Hint: Modify the solution to the previous exercise.

3. (Stein & Shakarchi, Chapter 4 Exercise 3.) Show, by contour integration, that

   $$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2}\,\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x = \mathrm{e}^{-2\pi a|\xi|}$$

   for $a > 0$ and $\xi \in \mathbb{R}$. Then check explicitly that

   $$\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi a|\xi|}\,\mathrm{e}^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi = \frac{1}{\pi}\frac{a}{a^2+x^2}\,.$$

4. (Laurent series expansion, cf. Stein & Shakarchi, Chapter 3 Problem 3.) Let f be holomorphic on an open set $\Omega \subset \mathbb{C}$ containing the closed annulus $\{z\colon r_1 \leq |z| \leq r_2\}$ where $0 < r_1 < r_2$. Prove that f has a series representation

   $$f(z) = \sum_{n=-\infty}^{\infty} a_n\,z^n$$

   which converges absolutely in the interior of the annulus.

Hint: If z is in the interior of the annulus, you can use the Cauchy integral formula to write

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{C_{r_2}(0)} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \int_{C_{r_1}(0)} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta\,.$$

Then, use an appropriate geometric series expansion for $1/(\zeta - z)$. (Similar tricks are used to prove that holomorphic implies analytic and appear in the proof of the Poisson summation formula.)