Complex Analysis

Homework 8

Due in class Thursday, November 18, 2021

1. Recall from class that f is an automorphism of the unit disc $\mathbb{D}$ if and only if there exist $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that
$$f(z) = \mathrm{e}^{\mathrm{i}\theta} \, \frac{\alpha - z}{1 - \overline{\alpha} z} \, .$$
Pulling back this automorphism of the disc to the upper half plane via
$$\psi = F^{-1} \circ f \circ F \, ,$$
where $F \colon \mathbb{H} \to \mathbb{D}$ is the conformal map
$$F(z) = \frac{\mathrm{i} - z}{\mathrm{i} + z} \, , \qquad F^{-1}(w) = \mathrm{i} \, \frac{1 - w}{1 + w} \, ,$$
show that
$$\psi(z) = \frac{az + b}{cz + d}$$
where a, b, c, d are real and satisfy $ad - bc = 1$.

   Hint: The computations are tedious, you may want to use a computer algebra system. Alternatively, look at the hints in Stein & Shakarchi, Chapter 8, Exercise 16 which provide a structured way of doing this argument by hand.

2. Show that the exponential $f(z) = \mathrm{e}^z$ is a conformal map from the strip
$$\Omega = \{0 < \operatorname{Im} z < b\}$$
to a sector
$$S = \{0 < \arg z < b\}$$
provided $0 < b < 2\pi$.

3. Show that the Joukowski map
$$f(z) = \frac{1}{2} \left(z + \frac{1}{z} \right)$$
is a conformal map from the exterior of the unit disk $\{z \colon |z| > 1\}$ to the exterior of the segment of the real axis $[-1, 1]$.

4. (Stein & Shakarchi, Chapter 8 Exercise 11.) Show that if $f\colon D_R(0) \to \mathbb{C}$ is holomorphic, with $|f(z)| \le M$ for some $M > 0$, then

$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)}\, f(z)} \right| \le \frac{|z|}{MR} .$$

Hint: Use the Schwarz lemma.