Functional Analysis

Final Exam

May 20, 2009

1. (a) Let E be a Banach space with closed subspace $G \subsetneq E$. Show that G is nowhere dense in E.

   (b) Let V be a vector space with a countably infinite (algebraic) basis B, i.e. every $x \in V$ can be represented as a finite linear linear combination of elements from V. Show that E cannot be complete with respect to *any* norm.

2. Let E be a Banach space and $A \in \mathcal{L}(E)$. Show that $\sigma(A)$ is compact.

3. Let H be a Hilbert space and A a linear, closed, densely defined, skew-adjoint (i.e. $A^* = -A$), unbounded operator on H. Show that

   (a) $\sigma(A) \subset i\mathbb{R}$;

   (b) A has no residual spectrum.

4. Let $H = \ell^2(\mathbb{N})$ and define an operator A via $Ae_n = n^{-1} e_n$ for $n \in \mathbb{N}$, where $\{e_n\}$ is the canonical basis in ℓ^2. Further, let L be the left shift operator on ℓ^2, i.e. $L(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots)$.

   (a) Show that LA is compact.

   (b) What is $\sigma(LA)$?

5. Let E, F be Banach spaces and $A \colon \mathcal{D}(A) \subset E \to F$ be a linear, closed, densely defined, and invertible unbounded operator. Show that A^* is invertible with inverse $(A^{-1})^*$.

6. Consider $A = \partial_x$ on $L^2([0,1])$ with

$$\mathcal{D}(A) = \{u \in L^2 \colon \partial_x u \in L^2([0,1]), u(0) = 0\}.$$

   (a) Show that $A^* = -\partial_x$ with

$$\mathcal{D}(A^*) = \{u \in L^2 \colon \partial_x u \in L^2([0,1]), u(1) = 0\}.$$

   (b) Describe the point, continuous, and residual spectrum.