Functional Analysis

Homework 5

due March 20, 2009

1. Give an example of a Banach space E having two closed subspaces G and L such that $G + L$ is not closed.

2. Give an example of a Banach space E with subspace N such that $\overline{N} \neq (N^\perp)^\perp$.

   (See hints in lecture notes, p. 21 top paragraph.)

3. Let E be a Banach space and G a closed subspace. Prove that

$$\operatorname{dist}(f, G^\perp) = \sup_{\substack{x \in G \\ \|x\| \leq 1}} f(x)$$

   for any $f \in E^*$.

4. Let $A\colon \mathcal{D}(A) \subset E \to F$ be an unbounded linear operator. Recall that the *graph* of A is defined

$$\Gamma(A) = \{(u, Au)\colon u \in \mathcal{D}(A)\},$$

   the *kernel* or *nullspace* of A is

$$\operatorname{Ker}(A) = \{u \in \mathcal{D}(A)\colon Au = 0\},$$

   and the *image* or *range* of A is

$$\operatorname{Range}(A) = \{Au\colon u \in \mathcal{D}(A)\}.$$

   Now let $X = E \times F$ with subspaces $G = \Gamma(A)$ and $L = E \times \{0\}$. Verify the following.

   (a) $\operatorname{Ker}(A) \times \{0\} = G \cap L$,

   (b) $E \times \operatorname{Range}(A) = G + L$,

   (c) $\{0\} \times \operatorname{Ker}(A^*) = G^\perp \cap L^\perp$,

   (d) $\operatorname{Range}(A^*) \times F^* = G^\perp + L^\perp$.