Functional Analysis

Homework 6

due April 1, 2009

1. Let E, F be Banach spaces and $A\colon \mathcal{D}(A) \subset E \to F$ an unbounded operator. Show that the following are equivalent.

   (a) A is closeable. (I.e., A has a closed extension.)

   (b) $\overline{\Gamma(A)}$ is the graph of a closed operator $\overline{A}$, the smallest closed extension of A.

   (c) $(0, v) \in \overline{\Gamma(A)}$ implies that $v = 0$.

2. Let E be a Banach space and A, B operators from E into itself with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$.

   (a) What is $\mathcal{D}(AB)$?

   (b) Show that A closed and B bounded implies AB closed.

   (c) Show that A bounded and B closed does not imply that AB is closed.

3. Consider ∂_r as an unbounded operator on $L^2((0,\infty), r^2\,\mathrm{d}r)$ with domain

$$\mathcal{D}(\partial_r) = \{u \in L^2((0,\infty), r^2\,\mathrm{d}r)\colon u \in AC((0,\infty)), \partial_r u \in L^2((0,\infty), r^2\,\mathrm{d}r)\}\,.$$

   (a) Show that $r^{1/2}\,u(r) \to 0$ as $r \to \infty$ for every $u \in \mathcal{D}(\partial_r)$.

   (b) Compute the adjoint of ∂_r and its domain.

4. Consider ∂_x acting on $L^2(\mathbb{T})$ with

$$\mathcal{D}(\partial_x) = \{u \in L^2(\mathbb{T})\colon u \in AC(\mathbb{T}), \partial_x u \in L^2(\mathbb{T})\}\,.$$

   (a) Show that $\partial_x^* = -\partial_x$ on $\mathcal{D}(\partial_x)$ and that $\mathcal{D}(\partial_x^*) \supset \mathcal{D}(\partial_x)$.

   (b) Find $\mathrm{Range}(\partial_x)$ and $\mathrm{Ker}(\partial_x)$, then conclude that $\mathcal{D}(\partial_x^*) = \mathcal{D}(\partial_x)$.