Jacobs University
School of Engineering and Science

Spring Term 2010

Problem Set 3

Stochastic Processes

Homework Problems

3.1. Suppose you have a biased coin toss experiment (probability for heads up is $0 < p < 1$), and you repeat it n times, where $n \geq 1$ is arbitrary. Let p_n denote the probability that the number of heads is even. Using conditional probabilities, find a recursion for p_n, and then give an explicit formula for p_n. Give an interpretation of what happens with p_n if $n \to \infty$ resp. $p \to 1$ resp. $p \to 0$.

3.2. Suppose Mike is interested in obtaining a good which costs N currency units but owns only k units ($0 < k < N$). Since he does not have credit with a bank, he goes to his rich friend and gambles with him in the following way: They toss a fair coin, if heads comes up the friend gives him a currency unit, otherwise he pays a unit to his friend. Then they continue until the game or ends successfully for Mike when he reaches N units, or it ends in bankruptcy for Mike if he has no units left. Compute the probability p_k that Mike eventually ends in bankruptcy.
Hint: Use conditioning with respect to B and $\bar{B}$, where B is the event that in the first toss heads comes up. This leads to a recursion involving p_{k-1}, p_k, and p_{k+1}.

3.3. Suppose that ξ is a random variable with distribution function F_ξ. For any interval $I = (a, b]$ with $F_\xi(b) - F_\xi(a) > 0$, find a formula for $E(\xi|\xi \in I)$.

3.4. Prove the following property: If ξ is a random variable on $(\Omega, \mathcal{F}, P)$, $\mathcal{G} \subset \mathcal{F}$, and $\psi : \Omega \to \mathbb{R}$ a discrete random variable with values y_i such that $\{\psi = y_i\} \in \mathcal{G}$, $i = 1, \ldots, n$, then

$$E(\psi\xi|\mathcal{G}) = \psi E(\xi|\mathcal{G}).$$

3.5. Do another example of finding a Borel function $F : \mathbb{R} \to \mathbb{R}$ that describes the conditional expectation in the form $E(\xi|\eta) = F(\eta)$ (the existence of such an F follows from the Doob-Dynkin lemma) if the joint probability density $f_{\xi,\eta}(x, y)$ of ξ, η is given. Choose $f_{\xi,\eta}(x, y) = e^{-2x-y/2}$ if $x, y \geq 0$, and $f_{\xi,\eta}(x, y) = 0$ otherwise. Interpret the result.
Hint: Study Exercises 2.7, 2.8, and 2.16 in the textbook. If you are not familiar with joint distributions and densities (see Definition 1.8 ff), take any other textbook to familiarize yourself with these notions.
Note: Since I did not make this clear enough during class: The value $F(y)$ is interpreted as the expectation of ξ with respect to the event $\{\omega : \eta(\omega) = y\}$, in short

$$E(\xi|\eta = y) := F(y),$$

independently on whether $P(\eta = y) > 0$ or not. Check that this is in agreement with the definition of $E(\xi|B)$ in Chapter 2.1, what is $F(y)$ for the situation discussed in Chapter 2.2? The same way one can define the *conditional probability of an event A given* η, i.e., the Borel function p_A satisfying

$$P(A|\eta) := E(\chi_A|\eta) = p_A(\eta).$$

Due Date: Thursday March 11, in class.