General Mathematics and ACM II

Final Exam

May 21, 2011

1. Show that in a finite graph without cycles, there is a vertex of valency at most one. (10)

2. Let ℓ be a fixed line in the plane. Recall that a *glide reflection* with axis ℓ is a transformation $U = R_\ell \Pi$ where R_ℓ is the line reflection about ℓ and Π is some nonidentity translation which leaves ℓ invariant.

   (a) Show that $R_\ell \Pi = \Pi R_\ell$.

   (b) Show that $U^{-1} = R_\ell \Pi^{-1}$.

   (c) Show that U^{-1} is a glide reflection with axis ℓ.

   (d) Consider the set of all glide reflections with axis ℓ. Is it a group? If not, describe the group it generates.

   (5+5+5+5)

3. Fix $N \in \mathbb{N}$. The ith *circular shift matrix* is the $N \times N$ matrix

$$S_i = \begin{pmatrix} & & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & 0 & 1 & 0 \\ 0 & & & & & & 0 & 1 \\ 1 & 0 & & & & & & \vdots \\ 0 & 1 & 0 & & & & & \\ \vdots & \ddots & \ddots & \ddots & & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & & \end{pmatrix}$$

where the leftmost 1 appears in the ith row.

Further, let M denote the diagonal matrix

$$M = \begin{pmatrix} m_1 & 0 & & & \\ 0 & m_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & 0 & m_N \end{pmatrix}$$

where $|m_1| = \cdots = |m_N| = 1$.

(a) Under which condition on the values $m_1, \ldots, m_N$ does $S_i M = M S_i$ hold for every $i = 1, \ldots, N$?

(b) Show that $\{(S_2 M)^i \colon i \in \mathbb{Z}\}$ is a group. What is its order (the number of elements)?

(c) There is a correspondence of this construction with the Kac ring model. Explain!

(d) There is a correspondence of this construction with glide reflections. Explain!

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4. Solve the linear programming problem

$$\text{minimize } z = x + 3\,y$$

subject to

$$\begin{aligned} x + 2\,y &\geq 2\,, \\ 2\,x + y &\geq 2\,, \\ x &\geq 0\,, \\ y &\geq 0\,, \end{aligned}$$

using either the graphical method or the simplex method. (10)

5. The linear programming problem

$$\begin{aligned} &\text{maximize } \boldsymbol{c}^T \boldsymbol{x} \\ &\text{subject to } A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{P}$$

has a corresponding symmetric dual problem

$$\begin{aligned} &\text{minimize } \boldsymbol{b}^T \boldsymbol{y} \\ &\text{subject to } A^T \boldsymbol{y} \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\,. \end{aligned} \tag{D}$$

Suppose that $\boldsymbol{x}$ is feasible for (P) and $\boldsymbol{y}$ is feasible for (D).

(a) Show that $\boldsymbol{c}^T\boldsymbol{x} \leq \boldsymbol{b}^T\boldsymbol{y}$.

(b) Conclude that if $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{y}$, then $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal for their respective linear programming problems.

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6. Recall that for $v \in \mathbb{C}^N$, the discrete Fourier transform of v is defined

$$\hat{v}_k = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}jkh} \, v_j$$

with $h = 2\pi/N$ and for $k = 0, \ldots, N-1$.

(a) Set $w_j = \mathrm{e}^{\mathrm{i}jmh} \, v_j$ for $j = 1, \ldots, N$. Express the discrete Fourier transform of w in terms of the discrete Fourier transform of v.

(b) Let $v \in \mathbb{R}^N$ be a vector of *real* numbers. Show that its discrete Fourier transform satisfies

$$\overline{\hat{v}_k} = \hat{v}_{N-k} \, ,$$

where the overbar denotes the complex conjugate.

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