1. Show that in a finite graph without cycles, there is a vertex of valency at most one. (10)

Solution 1: Let $\langle v_1, \dots, v_k \rangle$ be a chain of maximal length. (Which exists since the graph is finite.) Then v_k is a vertex of valency at most one, for if not, the chain would either not be maximal, or there would have to be a cycle.

Solution 2: Take Euler's formula applied to a connected component of the graph. Since there are no cycles, $|F| = 1$, so

$$|V| - |E| = 1$$

But $\sum_{v \in G} \delta(v) = 2|E| = 2|V| - 2$

So there must be at least two vertices with valency no more than 1.

2. Let ℓ be a fixed line in the plane. Recall that a *glide reflection* with axis ℓ is a transformation $\mathrm{U} = \mathrm{R}_\ell \Pi$ where R_ℓ is the line reflection about ℓ and Π is some nonidentity translation which leaves ℓ invariant.

(a) Show that $\mathrm{R}_\ell \Pi = \Pi \mathrm{R}_\ell$.

(b) Show that $\mathrm{U}^{-1} = \mathrm{R}_\ell \Pi^{-1}$.

(c) Show that U^{-1} is a glide reflection with axis ℓ.

(d) Consider the set of all glide reflections with axis ℓ. Is it a group? If not, describe the group it generates.

(5+5+5+5)

(a) WLOG let ℓ be the x-axis and let Π be right translation by one unit. Then

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{\Pi} \begin{pmatrix} x+1 \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{R_\ell} \begin{pmatrix} x \\ -y \end{pmatrix}$$

It is then trivial to check that

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{R_\ell \Pi} \begin{pmatrix} x+1 \\ -y \end{pmatrix} \quad \text{and so does } \Pi R_\ell$$

(A geometric argument, looking at the image of a point under the various maps is equally easy.)

(b) $$R_\ell \Pi^{-1} U = R_\ell \Pi^{-1} R_\ell \Pi \overset{(a)}{=} R_\ell \Pi^{-1} \Pi R_\ell = R_\ell^2 = Id$$

$$\Rightarrow R_\ell \Pi^{-1} = U^{-1}.$$

(c) Since T^{-1} is also a non-identity translation which leaves ℓ invariant, by (b), $U^{-1} = R_\ell T^{-1}$ satisfies the definition of a glide reflection.

(d) The set of glide reflections with axis ℓ does not contain the identity, so is not a group.

The group generated by it contains all translations along ℓ (the composition of two glide reflections), hence also reflection about ℓ ($R_\ell = U \circ T^{-1}$!).

Thus, it is homeomorphic to the group $\mathbb{R} \times \mathbb{Z}_2$.

3. Fix $N \in \mathbb{N}$. The ith *circular shift matrix* is the $N \times N$ matrix

$$S_i = \begin{pmatrix} & & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & 0 & 1 & 0 \\ 0 & & & & & & 0 & 1 \\ 1 & 0 & & & & & & \vdots \\ 0 & 1 & 0 & & & & & \\ \vdots & \ddots & \ddots & \ddots & & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & & \end{pmatrix}$$

where the leftmost 1 appears in the ith row.

Further, let M denote the diagonal matrix

$$M = \begin{pmatrix} m_1 & 0 & & & \\ 0 & m_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & 0 & m_N \end{pmatrix}$$

where $|m_1| = \cdots = |m_N| = 1$.

(a) Under which condition on the values $m_1, \ldots, m_N$ does $S_i M = M S_i$ hold for every $i = 1, \ldots, N$?

(b) Show that $\{(S_2 M)^i \colon i \in \mathbb{Z}\}$ is a group. What is its order (the number of elements)?

(c) There is a correspondence of this construction with the Kac ring model. Explain!

(d) There is a correspondence of this construction with glide reflections. Explain!

(5+5+5+5)

(a) $S_i M = \begin{pmatrix} & m_{N-i+2} & & \\ & & \ddots & \\ & & & m_N \\ m_1 & & & \\ & \ddots & & \\ & & m_{N-i+1} & \end{pmatrix}$, $M S_i = \begin{pmatrix} & m_1 & & \\ & & \ddots & \\ & & & m_{i-1} \\ m_i & & & \\ & \ddots & & \\ & & m_N & \end{pmatrix}$

Since $m_j = \pm 1$, identity holds for all $i = 1, \ldots, N$ if and only if all the m_j are the same.

(b)

$$S_2 M = \begin{pmatrix} 0 & \cdots & \cdots & 0 & m_N \\ m_1 & & & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & m_{N-1} & 0 \end{pmatrix}$$

$$(S_2 M)^2 = \begin{pmatrix} 0 & \cdots & \cdots & 0 & m_{N-1}m_N & 0 \\ 0 & & & & & m_N m_1 \\ m_1 m_2 & & & & & 0 \\ & \ddots & & & & \vdots \\ & & m_{N-2}m_{N-1} & 0 & & 0 \end{pmatrix}$$

$$\vdots$$

$$(S_2 M)^N = \begin{pmatrix} m_1 m_2 \cdots m_N & & 0 \\ & \ddots & \\ 0 & & m_1 m_2 \cdots m_N \end{pmatrix} = \begin{cases} I & \text{if } m_1 \cdots m_N = 1 \\ -I & \text{if } m_1 \cdots m_N = -1 \end{cases}$$

So if $m_1 \cdots m_N = 1$, the set $G = \{(S_2 M)^i : i \in \mathbb{Z}\}$ contains N elements, if $m_1 \cdots m_N = -1$, it contains $2N$ elements. Further, $I \in G$.

By the rules of matrix exponentiation, identifying the element $(S_2 M)^i \in G$ with $i \in \mathbb{Z}_N$ (resp. $\mathbb{Z}_{2N}$), we see that this defines a group homomorphism with $\mathbb{Z}_N$ (resp. $\mathbb{Z}_{2N}$), hence G is a group.

(Note: it is straightforward, but tedious, to verify the group axioms one-by-one.)

(c) Let $x_i(t)$ denote the color of the i-th site, with 1 for black and -1 for white, and let $m_i = -1$ iff the i-th edge carries a marker. Then the evolution of the ring is given by

$$\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{pmatrix} = (S_2 M)^t \begin{pmatrix} x_1(0) \\ \vdots \\ x_N(0) \end{pmatrix},$$

i.e., G could be called the "Kac ring group".

(d) If $m_1 = \dots = m_N = -1$, then $S_2 M$ represents a (cyclic) shift and reflection. Moreover, $S_i M$ satisfies the same commutator relation as a glide reflection, so it could be seen as a discrete glide reflection.

In this sense, the Kac ring evolution is a generalised glide reflection where some sites are reflected and others are not.

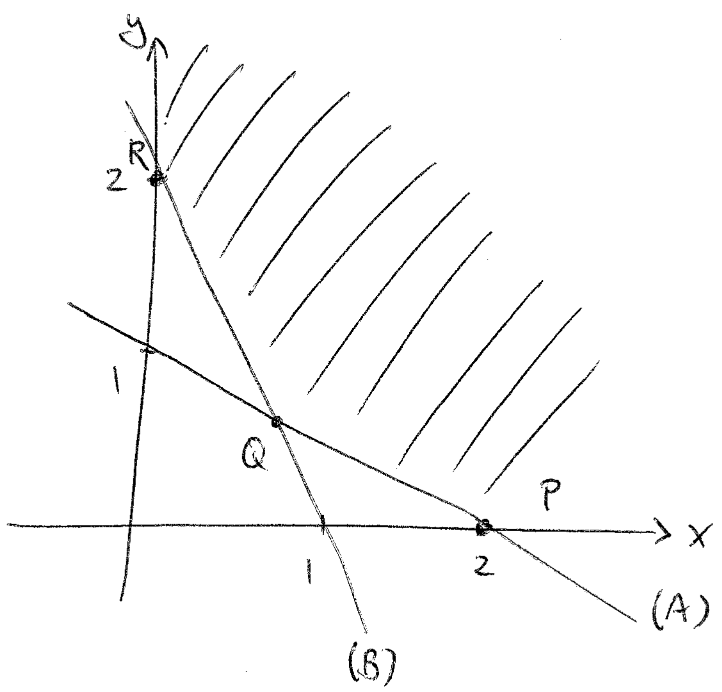
4. Solve the linear programming problem

$$\text{minimize } z = x + 3y$$

subject to

$$\begin{aligned} x + 2y &\geq 2, \qquad (A) \\ 2x + y &\geq 2, \qquad (B) \\ x &\geq 0, \\ y &\geq 0, \end{aligned}$$

using either the graphical method or the simplex method. (10)



Compute coordinates of Q:

$$\left(\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -3 & -2 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} \end{array}\right)$$

$$\Rightarrow \quad Q = \left(\tfrac{2}{3}, \tfrac{2}{3}\right), \quad z = \tfrac{2}{3} + 2$$

$$P = (2, 0), \quad z = 2$$

$$R = (0, 2), \quad z = 6$$

$\Rightarrow$ The minimum is attained at P with $z = 2$.

5. The linear programming problem

$$\begin{aligned} &\text{maximize } \mathbf{c}^\mathsf{T}\mathbf{x} \\ &\text{subject to } A\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge \mathbf{0} \end{aligned} \tag{P}$$

has a corresponding symmetric dual problem

$$\begin{aligned} &\text{minimize } \mathbf{b}^\mathsf{T}\mathbf{y} \\ &\text{subject to } A^\mathsf{T}\mathbf{y} \ge \mathbf{c}, \mathbf{y} \ge \mathbf{0}\,. \end{aligned} \tag{D}$$

Suppose that $\mathbf{x}$ is feasible for (P) and $\mathbf{y}$ is feasible for (D).

(a) Show that $\mathbf{c}^\mathsf{T}\mathbf{x} \le \mathbf{b}^\mathsf{T}\mathbf{y}$.

(b) Conclude that if $\mathbf{c}^\mathsf{T}\mathbf{x} = \mathbf{b}^\mathsf{T}\mathbf{y}$, then $\mathbf{x}$ and $\mathbf{y}$ are optimal for their respective linear programming problems.

(5+5)

(a) $c^T x \overset{(D)}{\le} (A^T y)^T x = y^T A x \overset{(P)}{\le} y^T b = b^T y$

(b) Since $b^T y$ is an upper bound for the objective function of (P), attainment of the upper bound implies optimality for (P). Likewise for (D).

6. Recall that for $v \in \mathbb{C}^N$, the discrete Fourier transform of v is defined

$$\hat{v}_k = \frac{1}{N} \sum_{j=0}^{N-1} e^{-ijkh} v_j$$

with $h = 2\pi/N$ and for $k = 0, \ldots, N-1$.

(a) Set $w_j = e^{ijmh} v_j$ for $j = 1, \ldots, N$. Express the discrete Fourier transform of w in terms of the discrete Fourier transform of v.

(b) Let $v \in \mathbb{R}^N$ be a vector of *real* numbers. Show that its discrete Fourier transform satisfies

$$\overline{\hat{v}_k} = \hat{v}_{N-k},$$

where the overbar denotes the complex conjugate.

(5+5)

(a) $$\hat{w}_k = \frac{1}{N} \sum_{j=0}^{N-1} \underbrace{e^{-ijkh} e^{ijmh}}_{= e^{-ij(k-m)h}} v_j = \hat{v}_{k-m}$$

(b) $$\overline{\hat{v}_k} = \overline{\frac{1}{N} \sum_{j=0}^{N-1} e^{-ijkh} v_j} = \frac{1}{N} \sum_{j=0}^{N-1} e^{ijkh} v_j = \hat{v}_{-k}$$

Since $\hat{v}_k$ is periodic with period N, we can write

$$\hat{v}_{-k} = \hat{v}_{N-k}$$

to translate back to the standard range of wave numbers.